

REGULARITY OF MIXED SPLINE SPACES

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ABSTRACT. We derive bounds on the regularity of the algebra $C^\alpha(\mathcal{P})$ of mixed splines over a central polytopal complex $\mathcal{P} \subset \mathbb{R}^3$. As a consequence we bound the largest integer d (the postulation number) for which the Hilbert polynomial $HP(C^\alpha(\mathcal{P}), d)$ disagrees with the Hilbert function $HF(C^\alpha(\mathcal{P}), d) = \dim C^\alpha(\mathcal{P})_d$. The polynomial $HP(C^\alpha(\mathcal{P}), d)$ has been computed in [12], building on [16, 20]. Hence the regularity bounds obtained indicate when a known polynomial gives the correct dimension of the spline space $C^\alpha(\mathcal{P})_d$. In the simplicial case with all smoothness parameters equal, we recover a bound originally due to Hong [18] and Ibrahim and Schumaker [19].

1. INTRODUCTION

Let \mathcal{P} be a subdivision of a region in \mathbb{R}^n by convex polytopes. $C^r(\mathcal{P})$ denotes the set of piecewise polynomial functions (splines) on \mathcal{P} that are continuously differentiable of order r . Splines are a fundamental tool in approximation theory and numerical analysis [9]; more recently they have also appeared in a geometric context, describing the equivariant cohomology ring of toric varieties [24]. Practical applications include surface modelling, computer-aided design, and computer graphics [9].

One of the fundamental questions in spline theory is to determine the dimension of the space $C_d^r(\mathcal{P})$ of splines of degree at most d . In the bivariate, simplicial case, these questions are studied by Alfed-Schumaker in [2] and [3] using Bernstein-Bezier methods. A signature result in [3] is a formula for $\dim C_d^r(\Delta)$ when $d \geq 3r + 1$ and $\Delta \subset \mathbb{R}^2$ is a generic simplicial complex. For $\Delta \subset \mathbb{R}^2$ simplicial and nongeneric, Hong [18] and Ibrahim-Schumaker [19] derive a formula for $\dim C_d^r(\Delta)$ when $d \geq 3r + 2$ as a byproduct of constructing local bases for these spaces.

An algebraic approach to the dimension question was pioneered by Billera in [6] using homological and commutative algebra. In [7], Billera-Rose show that $C_d^r(\mathcal{P}) \cong C^r(\hat{\mathcal{P}})_d$, the d th graded piece of the algebra $C^r(\hat{\mathcal{P}})$ of splines on the cone $\hat{\mathcal{P}}$ over \mathcal{P} . The function $HF(C^r(\hat{\mathcal{P}}), d) = \dim_{\mathbb{R}} C^r(\hat{\mathcal{P}})_d$ is known as the *Hilbert function* of $C^r(\hat{\mathcal{P}})$ in commutative algebra, and a standard result is that the values of the Hilbert function eventually agree with the *Hilbert polynomial* $HP(C^r(\hat{\mathcal{P}}), d)$ of $C^r(\hat{\mathcal{P}})$. An important invariant of $C^r(\hat{\mathcal{P}})$ is the *postulation number* $\wp(C^r(\hat{\mathcal{P}}))$, which is the largest integer d so that $HP(C^r(\hat{\mathcal{P}}), d) \neq HF(C^r(\hat{\mathcal{P}}), d)$. In this

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Analytic Methods		
Bound	Context	Computed by
$\wp(C^r(\hat{\Delta})) \leq 3r$	<i>generic</i> simplicial $\Delta \subset \mathbb{R}^2$	Alfeld-Schumaker [3]
$\wp(C^r(\hat{\Delta})) \leq 3r + 1$	<i>all</i> simplicial $\Delta \subset \mathbb{R}^2$	Hong [18] Ibrahim-Schumaker [19]
$\wp(C^1(\hat{\Delta})) \leq 3$	<i>all</i> simplicial $\Delta \subset \mathbb{R}^2$	Alfeld-Piper-Schumaker [1]
$\wp(C^1(\hat{\Delta})) \leq 7$	<i>generic</i> simplicial $\Delta \subset \mathbb{R}^3$	Alfeld-Schumaker-Whiteley [4]
Homological Methods		
Bound	Context	Computed by
$\wp(C^r(\hat{\Delta})) \leq 4r$	<i>all</i> simplicial $\Delta \subset \mathbb{R}^2$	Mourrain-Villamizar [23]
$\wp(C^1(\hat{\Delta})) \leq 1$	<i>generic</i> simplicial $\Delta \subset \mathbb{R}^2$	Billera [6]

TABLE 1. Bounds on $\wp(C^r(\hat{\Delta}))$

terminology the Alfeld-Schumaker result above could be viewed as a computation of $HP(C^r(\hat{\Delta}), d)$ plus the bound $\wp(C^r(\hat{\Delta})) \leq 3r$.

The goal of this paper is to provide upper bounds on the postulation number $\wp(C^\alpha(\mathcal{P}))$ for *central* polytopal complexes $\mathcal{P} \subset \mathbb{R}^{n+1}$, where $C^\alpha(\mathcal{P})$ is the algebra of *mixed splines* over \mathcal{P} . A *central* polytopal complex is one in which the intersection of all interior faces is nonempty; if \mathcal{P} is central then splines on \mathcal{P} are a graded algebra. *Mixed splines* are splines in which different smoothness conditions are imposed across codimension one faces.

The main reason for bounding $\wp(C^\alpha(\mathcal{P}))$ is that the Hilbert polynomial of $C^\alpha(\mathcal{P})$ has been computed in situations where there are no known bounds on $\wp(C^\alpha(\mathcal{P}))$, rendering these formulas impractical. Currently, bounds which do not make heavy restrictions on the complex \mathcal{P} are known only in the simplicial case. These bounds are recorded in Table 1. For particular types of complexes \mathcal{P} , better and sometimes exact bounds are known for $\wp(C^r(\mathcal{P}))$. In contrast, the Hilbert polynomial $HP(C^\alpha(\mathcal{P}), d)$ has been computed for *all* central polytopal complexes $\mathcal{P} \subset \mathbb{R}^3$. This is done in the simplicial case with mixed smoothness by Schenck-Geramita [16], in the polytopal case with uniform smoothness by Schenck-McDonald [20], and in the polytopal case with mixed smoothness and boundary conditions, by the author [12]. In this paper we provide the first bound on $\wp(C^\alpha(\mathcal{P}))$ for all central polytopal complexes $\mathcal{P} \subset \mathbb{R}^3$. Specifically, given *smoothness parameters* $\alpha(\tau)$ associated to each codimension one face $\tau \in \mathcal{P}$, our first result is the following.

Theorem 6.7 Let $\mathcal{P} \subset \mathbb{R}^3$ be a central, pure, hereditary three-dimensional polytopal complex. Set

$$e(\mathcal{P}) = \max_{\tau \in \mathcal{P}_2^0} \left\{ \sum_{\gamma \in (\text{st}(\tau))_2} (\alpha(\gamma) + 1) \right\},$$

where $\text{st}(\tau)$ denotes the star of τ and $(\text{st}(\tau))_2$ denotes the 2-faces of $\text{st}(\tau)$. Then

$$\wp(C^\alpha(\mathcal{P})) \leq e(\mathcal{P}) - 3.$$

In particular, $HP(C^\alpha(\mathcal{P}), d) = \dim_{\mathbb{R}} C^\alpha(\mathcal{P})_d$ for $d \geq e(\mathcal{P}) - 2$.

From an algebraic perspective, another reason for bounding $\wp(C^\alpha(\mathcal{P}))$ is that almost all existing bounds, including most in Table 1, have been computed using

analytic techniques. There are a few instances where algebraic techniques are applied to bound $\wp(C^\alpha(\mathcal{P}))$. In [6], Billera proves $\wp(C^1(\hat{\Delta})) \leq 1$ for *generic* simplicial complexes (this result relies on a computation of Whiteley [29]). The most general bound produced by homological techniques to date is by Mourrain-Villamizar [23]; building on work of Schenck-Stillman [27] they prove that $\wp(C^r(\hat{\Delta})) \leq 4r$ for Δ a planar simplicial complex, recovering an earlier result of Alfeld-Schumaker [2]. Our second result is the following.

Theorem 7.2 Let $\Delta \subset \mathbb{R}^3$ be a central, pure, hereditary three-dimensional simplicial complex. For a 2-face $\tau \in \Delta_2^0$, set

$$M(\tau) = (\alpha(\tau) + 1) + \max\{(\alpha(\gamma_1) + 1) + (\alpha(\gamma_2) + 1) \mid \gamma_1 \neq \gamma_2 \in (\text{st}(\tau))_2\}.$$

Then

$$\wp(C^\alpha(\Delta)) \leq \max_{\tau \in \Delta_2^0} \{M(\tau)\} - 2.$$

In particular, $HP(C^\alpha(\Delta), d) = \dim_{\mathbb{R}} C^r(\Delta)_d$ for $d \geq \max_{\tau \in \Delta_2^0} \{M(\tau)\} - 1$.

Setting $\alpha(\tau) = r$ for all $\tau \in \Delta_2^0$, we recover that $HP(C^r(\hat{\Delta}), d) = \dim C^r(\hat{\Delta})_d$ for $d \geq 3r + 2$. This was originally proved via constructing local bases by Hong [18] and Ibrahim-Schumaker [19], and is the best bound valid for all planar simplicial complexes recorded in Table 1.

A key tool we use to prove these results is the *Castelnuovo-Mumford regularity* of $C^\alpha(\mathcal{P})$, denoted $\text{reg}(C^\alpha(\mathcal{P}))$. The relationship between $\text{reg}(C^\alpha(\mathcal{P}))$ and $\wp(C^\alpha(\mathcal{P}))$ is discussed in detail in § 5. This invariant is also used by Schenck-Stillman in [26]. Our particular way of using regularity is inspired by an observation used in the Gruson-Lazarsfeld-Peskine theorem bounding the regularity of curves in projective space. In the context of splines this observation is roughly that, if we are lucky, we can bound $\text{reg}(C^\alpha(\mathcal{P}))$ by the regularity of a ‘bad’ approximation. This statement is made precise in Proposition 5.8 and Theorem 6.2. We take as our approximation certain locally-supported subalgebras of splines introduced in [11]. This could be viewed as an algebraic analogue of locally-supported bases used in [18, 19].

The paper is organized as follows. In § 2 we give some background on the spline algebra $C^\alpha(\mathcal{P})$, in particular the algebraic approach pioneered by Billera [6] and Billera and Rose [7]. We recall the construction of lattice-supported splines $LS^{\alpha,k}(\mathcal{P})$ introduced in [11]. These will provide approximations to $C^\alpha(\mathcal{P})$. In § 3 and § 4 we fit lattice-supported splines into a Čech-like complex. In § 5 we recall the definition of the regularity of a graded module and prove Proposition 5.8, which is our main tool for bounding regularity. In § 6 we prove our main results for bounding regularity of spline modules of low projective dimension and prove Theorem 6.7 bounding the regularity of $C^\alpha(\mathcal{P})$ where $\mathcal{P} \subset \mathbb{R}^3$ is a central polytopal complex. In § 7 we build on work of Tohaneanu-Minac [22] and prove the more precise regularity estimate for central simplicial complexes $\Delta \subset \mathbb{R}^3$ in Theorem 7.2. We close in § 9 with conjectured regularity bounds generalizing those derived in this paper. The two following examples illustrate our results.

1.1. Examples. Let $\mathcal{P} \subset \mathbb{R}^2$ be a subdivision of a topological 2-disk by convex polytopes. $C^r(\mathcal{P})$ is the algebra of r -splines on \mathcal{P} , where $\alpha(\tau) = r$ for every interior edge and $\alpha(\tau) = -1$ for every boundary edge. By Corollary 3.14 of [20], the Hilbert

polynomial of $C^r(\widehat{\mathcal{P}})$ is

$$(1) \quad HP(C^r(\widehat{\mathcal{P}}), d) = \frac{f_2}{2} d^2 + \frac{3f_2 - 2(r+1)f_1^0}{2} d + f_2 + \left(\binom{r}{2} - 1\right) f_1^0 + \sum_j c_j,$$

where f_i, f_i^0 are the number of i -faces and interior i -faces of \mathcal{P} , r is the smoothness parameter, and the constants c_j record the dimension of certain vector spaces coming from ideals of powers of linear forms.

Example 1.1. The complex \mathcal{Q} in Figure 1 has $f_2 = 4, f_1^0 = 6, f_3^0 = 3$. It is shown in § 4 of [20] that there are 4 constants c_j in the formula (1), and they are all equal to the constant

$$\binom{r+2}{2} + \left\lceil \frac{r+1}{2} \right\rceil \left(r - \left\lceil \frac{r+1}{2} \right\rceil \right)$$

Hence by equation (1),

$$(2) \quad HP(C^r(\widehat{\mathcal{Q}}), d) = 2d^2 - 6rd + 6\binom{r}{2} - 2 + 4\left(\binom{r+2}{2} + \left\lceil \frac{r+1}{2} \right\rceil \left(r - \left\lceil \frac{r+1}{2} \right\rceil \right)\right)$$

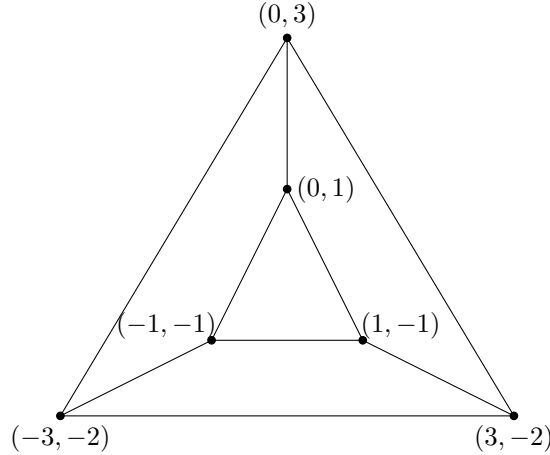


FIGURE 1. \mathcal{Q}

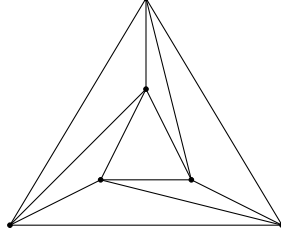
By Theorem 6.7, $\wp(C^r(\widehat{\mathcal{Q}})) \leq e(\mathcal{Q}) - 3$, where

$$e(\mathcal{Q}) = \max_{\tau \in \mathcal{P}_2^0} \left\{ \sum_{\gamma \in (\text{St}(\tau))_2} (\alpha(\gamma) + 1) \right\}.$$

The star of each interior edge of \mathcal{Q} has 5 edges which are interior. So $e(\mathcal{Q}) = 5(r+1)$ and the Hilbert function $HF(C^r(\widehat{\mathcal{Q}}), d)$ agrees with the Hilbert polynomial $HP(C^r(\widehat{\mathcal{Q}}), d)$ above for $d \geq 5(r+1) - 2$. Computations in Macaulay2 [17] suggest that $\wp(C^r(\widehat{\mathcal{Q}})) \leq 2(r+1) - 1$ (in fact the behavior is the same as Example 8.1 in § 8), indicating that there is room for improvement in Theorem 6.7.

In [16, Theorem 4.3], Geramita and Schenck give a formula for $HP(C^\alpha(\widehat{\Delta}), d)$, where Δ is a planar simplicial complex and α is the vector of smoothness parameters associated to codimension one faces.

Example 1.2. Triangulate the polytopal complex \mathcal{Q} in Example 1.1 to obtain the simplicial complex Δ below, with $f_2 = 7$, $f_1^0 = 9$, and $f_0^0 = 3$. Take smoothness parameters $\alpha(\tau) = 2$ on the edges of the center triangle and $\alpha(\tau) = 3$ on the six edges which connect interior vertices to boundary vertices. In Example 4.5 of [16],

FIGURE 2. Δ

Schenck and Geramita compute

$$HP(C^\alpha(\hat{\Delta}), d) = \binom{d+2}{2} - 3\binom{d-1}{2} + 3\binom{d-2}{2} + 6\binom{d-3}{2}.$$

By Theorem 7.2, $\wp(C^\alpha(\hat{\Delta})) \leq \max\{M(\tau) | \tau \in \Delta_1^0\} - 2$, where $M(\tau) = \alpha(\tau) + 1 + \max\{\alpha(\gamma_1) + 1 + \alpha(\gamma_2) + 1 | \gamma_1 \neq \gamma_2 \in (\text{st}(\tau))_1\}$. This yields $\wp(C^\alpha(\hat{\Delta})) \leq 10$, so the polynomial above gives the correct dimension of $C_d^\alpha(\Delta)$ for $d \geq 11$. Macaulay2 gives $\wp(C^\alpha(\hat{\Delta})) = 5$, so the formula is actually correct for $d \geq 6$.

2. SPLINES AND LATTICE-SUPPORTED SPLINES

We begin with some preliminary notions. A *polytopal complex* $\mathcal{P} \subset \mathbb{R}^n$ is a finite set of convex polytopes (called *faces* of \mathcal{P}) in \mathbb{R}^n such that

- If $\gamma \in \mathcal{P}$, then all faces of γ are in \mathcal{P} .
- If $\gamma, \tau \in \mathcal{P}$ then $\gamma \cap \tau$ is a face of both γ and τ (possibly empty).

The *dimension* of \mathcal{P} is the greatest dimension of a face of \mathcal{P} . The faces of \mathcal{P} are ordered via inclusion; a maximal face of \mathcal{P} is called a *facet* of \mathcal{P} , and \mathcal{P} is said to be *pure* if all facets are equidimensional. $|\mathcal{P}|$ denotes the underlying space of \mathcal{P} . \mathcal{P}_i and \mathcal{P}_i^0 denote the set of i -faces and the set of interior i -faces, respectively, while $f_i = |\mathcal{P}_i|$ and $f_i^0 = |\mathcal{P}_i^0|$. In the case that all facets of \mathcal{P} are simplices, \mathcal{P} is a simplicial complex and will be denoted by Δ . The boundary of \mathcal{P} , denoted $\partial\mathcal{P}$, is a polytopal complex of dimension $n - 1$, and is pure if \mathcal{P} is pure.

Given a complex \mathcal{P} and a face $\gamma \in \mathcal{P}$, the *star* of γ in \mathcal{P} , denoted $\text{st}_{\mathcal{P}}(\gamma)$, is defined by

$$\text{st}_{\mathcal{P}}(\gamma) := \{\psi \in \mathcal{P} | \exists \sigma \in \mathcal{P}, \psi \in \sigma, \gamma \in \sigma\}.$$

This is the smallest subcomplex of \mathcal{P} which contains all faces which contain γ . If the complex \mathcal{P} is understood we will write $\text{st}(\gamma)$.

For $\mathcal{P} \subset \mathbb{R}^n$, we define $G(\mathcal{P})$ to be the graph with a vertex for every facet (element of \mathcal{P}_n); two vertices are joined by an edge iff the corresponding facets σ and σ' satisfy $\sigma \cap \sigma' \in \mathcal{P}_{n-1}$. \mathcal{P} is said to be *hereditary* if $G(\text{st}_{\mathcal{P}}(\gamma))$ is connected for every nonempty $\gamma \in \mathcal{P}$. Throughout this paper, $\mathcal{P} \subset \mathbb{R}^n$ is assumed to be a pure, n -dimensional, hereditary polytopal complex.

Let $R = \mathbb{R}[x_1, \dots, x_n]$ be the polynomial ring in n variables, and $S = \mathbb{R}[x_0, \dots, x_n]$ the polynomial ring in $n+1$ variables. We will typically use R in inhomogeneous and S in homogeneous situations.

We now recall the definition of the ring of splines $C^r(\mathcal{P})$. For $U \subset \mathbb{R}^n$, let $C^r(U)$ denote the set of functions $F : U \rightarrow \mathbb{R}$ continuously differentiable of order r . For $F : |\mathcal{P}| \rightarrow \mathbb{R}$ a function and $\sigma \in \mathcal{P}_n$, F_σ denotes the restriction of F to σ . The module $C^r(\mathcal{P})$ of piecewise polynomials continuously differentiable of order r on \mathcal{P} is defined by

$$C^r(\mathcal{P}) := \{F \in C^r(|\mathcal{P}|) \mid F_\sigma \in R \text{ for every } \sigma \in \mathcal{P}_n\}$$

The polynomial ring R includes in $C^r(\mathcal{P})$ as globally polynomial functions (these are the *trivial* splines); this makes $C^r(\mathcal{P})$ an R -algebra via pointwise multiplication.

If \mathcal{P} is a hereditary complex, the global C^r condition can be expressed as a differentiability condition across internal faces of codimension one. For a codimension one face $\tau \in \mathcal{P}_{n-1}^0$, let l_τ denote a choice of affine form (unique up to scaling) which vanishes on τ . Then a function $F : |\mathcal{P}| \rightarrow \mathbb{R}$ which restricts to a polynomial on each facet is in $C^r(\mathcal{P})$ iff $l_\tau^{r+1} | (F_{\sigma_1} - F_{\sigma_2})$ for every pair of facets σ_1, σ_2 which intersect in a codimension one face τ [7].

In [16], Schenck and Geramita study the dimension of *mixed spline* spaces, in which the order of differentiability across codimension one faces varies. Specifically, let $\alpha = (\alpha(\tau) \mid \tau \in \mathcal{P}_{n-1})$ be a list of *smoothness parameters* $\alpha(\tau)$ associated to each codimension one face. We require that $\alpha(\tau) \geq 0$ for $\tau \in \mathcal{P}_{n-1}^0$ and $\alpha(\tau) \geq -1$ for $\tau \in (\partial\mathcal{P})_{n-1}$. Then the algebra $C^\alpha(\mathcal{P})$ of mixed splines on \mathcal{P} is defined as the set of splines $F \in C^0(\mathcal{P})$ satisfying

- $l_\tau^{\alpha(\tau)+1} | (F_{\sigma_1} - F_{\sigma_2})$ for $\tau \in \mathcal{P}_{n-1}^0$ with $\sigma_1 \cap \sigma_2 = \tau$
- $l_\tau^{\alpha(\tau)+1} | F_\sigma$ for $\tau \in (\partial\mathcal{P})_{n-1}$ with $\tau \in (\partial\sigma)_{n-1}$

For hereditary complexes, the usual ring of splines $C^r(\mathcal{P})$ is recovered by setting $\alpha(\tau) = r$ for every $\tau \in \mathcal{P}_{n-1}^0$ and $\alpha(\tau) = -1$ for every $\tau \in \mathcal{P}_{n-1}$. The following variant of [7, Proposition 4.3] encodes the mixed spline conditions as a matrix.

Lemma 2.1. *If \mathcal{P} is hereditary and $\alpha = (\alpha(\tau) \mid \tau \in \mathcal{P}_{n-1})$, $C^\alpha(\mathcal{P})$ fits into the graded exact sequence*

$$0 \rightarrow C^\alpha(\mathcal{P}) \rightarrow R^{f_n} \oplus \left(\bigoplus_{\tau \in \mathcal{P}_{n-1}} R(-\alpha(\tau) - 1) \right) \xrightarrow{\phi} R^{f_{n-1}} \rightarrow C \rightarrow 0$$

where $\phi = \begin{pmatrix} \delta_n & \begin{array}{c} l_{\tau_1}^{\alpha(\tau_1)+1} \\ \vdots \\ l_{\tau_k}^{\alpha(\tau_k)+1} \end{array} \end{pmatrix},$

$k = f_{n-1}$, $C = \text{coker } \phi$ and the matrix δ_n is the top dimensional cellular boundary map of \mathcal{P} .

Since our results apply in the context of mixed splines, we will use these throughout the paper.

Let $R_{\leq d}$ and R_d be the set of polynomials $f \in R$ of degree $\leq d$ and degree d , respectively. For $\mathcal{P} \subset \mathbb{R}^n$ we have a filtration of $C^\alpha(\mathcal{P})$ by \mathbb{R} -vector spaces

$$C_d^\alpha(\mathcal{P}) := \{F \in C^\alpha(\mathcal{P}) \mid F_\sigma \in R_{\leq d} \text{ for all facets } \sigma \in \mathcal{P}_n\}.$$

A polytopal complex \mathcal{P} is called a *central* complex if all interior codimension one faces share a common face. We will always assume the origin $\mathbf{0} \in \mathbb{R}^n$ is contained in this common face. For central complexes we make the assumption that $\alpha(\tau) = -1$ for all codimension one faces $\tau \in \mathcal{P}_{n-1}$ so that $\mathbf{0} \notin \text{aff}(\tau)$. Then the diagonal portion of the matrix ϕ in Lemma 2.1 consists of forms of degree $\alpha(\tau) + 1$ and $C^\alpha(\mathcal{P})$ is graded. The graded pieces are the vector spaces

$$C^\alpha(\mathcal{P})_d := \{F \in C^\alpha(\mathcal{P}) \mid F_\sigma \in R_d \text{ for all facets } \sigma \in \mathcal{P}_n\}.$$

Given $\mathcal{P} \subset \mathbb{R}^n$, the cone $\hat{\mathcal{P}} \subset \mathbb{R}^{n+1}$ over \mathcal{P} is formed by taking the join of $\mathbf{0} \in \mathbb{R}^{n+1}$ with $i(\mathcal{P})$, where $i : \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$ is defined by $i(a_1, \dots, a_n) = (1, a_1, \dots, a_n)$. This is clearly a central complex. If \mathcal{P} comes with smoothness parameters α , extend these to smoothness parameters $\hat{\alpha}$ on $\hat{\mathcal{P}}$ by assigning

- $\hat{\alpha}(\tau') = \alpha(\tau)$ for $\tau' \in \hat{\mathcal{P}}_n$ which is a cone over $\tau \in \mathcal{P}_{n-1}$ and
- $\hat{\alpha}(\tau') = -1$ for $\tau' \in \hat{\mathcal{P}}_n$ so that $\mathbf{0} \notin \text{aff}(\tau')$

Since this extension is natural we will abuse notation and drop the hat on α , denoting $C^{\hat{\alpha}}(\hat{\mathcal{P}})$ by $C^\alpha(\hat{\mathcal{P}})$. In practice one computes the algebra $C^{\hat{\alpha}}(\hat{\mathcal{P}})$ by replacing the polynomial ring R by S in Lemma 2.1 and homogenizing the entries of the matrix ϕ used to compute $C^\alpha(\mathcal{P})$. The following lemma is proved the same way as Theorem 2.6 of [7].

Lemma 2.2. *Let $\mathcal{P} \subset \mathbb{R}^n$ be a polytopal complex with smoothness parameters α . Then $C_d^\alpha(\mathcal{P}) \cong C_d^{\hat{\alpha}}(\hat{\mathcal{P}})$ as \mathbb{R} -vector spaces.*

2.1. Lattice-Supported Splines. In [11] certain subalgebras $LS^{r,k}(\mathcal{P}) \subset C^r(\mathcal{P})$ are constructed as approximations to $C^r(\mathcal{P})$. This construction carries over directly to mixed splines; we will denote the corresponding submodules by $LS^{\alpha,k}(\mathcal{P})$. We briefly summarize the construction. For a pure n -dimensional subcomplex $\mathcal{Q} \subset \mathcal{P}$, not necessarily hereditary, define

$$C_{\mathcal{Q}}^\alpha(\mathcal{P}) := \{F \in C^\alpha(\mathcal{P}) \mid F_\sigma = 0 \text{ for all } \sigma \in \mathcal{P}_n \setminus \mathcal{Q}_n\}.$$

Let $\mathcal{P}^{-1} \subset \partial\mathcal{P}$ denote the set of faces of \mathcal{P} which are contained in a codimension one face τ so that $\alpha(\tau) = -1$; this is a subcomplex of $\partial\mathcal{P}$.

Definition 2.3. Let $\mathcal{P} \subset \mathbb{R}^n$ be a polytopal complex and α a list of smoothness parameters.

- (1) For $\tau \in \mathcal{P}$ a face, $\text{aff}(\tau)$ denotes the affine span of τ .
- (2) $\mathcal{A}(\mathcal{P}, \mathcal{P}^{-1})$ denotes the hyperplane arrangement $\bigcup_{\substack{\tau \in \mathcal{P}_{n-1} \\ \alpha(\tau) \geq 0}} \text{aff}(\tau)$.
- (3) $L_{\mathcal{P}, \mathcal{P}^{-1}}$ denotes the intersection semi-lattice $L(\mathcal{A}(\mathcal{P}, \mathcal{P}^{-1}))$ of $\mathcal{A}(\mathcal{P}, \mathcal{P}^{-1})$.

The elements $W \in L(\mathcal{P}, \mathcal{P}^{-1})$ are called *flats*. These consist of the whole space \mathbb{R}^n , the hyperplanes $\{\text{aff}(\tau) \mid \alpha(\tau) \geq 0\}$, and all nonempty intersections of these, ordered with respect to reverse inclusion. The *rank* of a flat W , denoted $\text{rk}(W)$, is its codimension as a vector space.

To each flat $W \in L(\mathcal{P}, \mathcal{P}^{-1})$ we associate a *lattice complex* \mathcal{P}_W as follows. Form a graph $G_W(\mathcal{P})$ whose vertices correspond to facets which have a codimension one face τ so that $W \subseteq \text{aff}(\tau)$. Connect two vertices corresponding to facets σ_1, σ_2 if $\sigma_1 \cap \sigma_2$ is a codimension one face of both and $W \subseteq \text{aff}(\sigma_1 \cap \sigma_2)$. Each connected component $G_W^i(\mathcal{P})$ of $G_W(\mathcal{P})$ is the dual graph of a unique subcomplex \mathcal{P}_W^i . The

lattice complex \mathcal{P}_W is defined as the disjoint union of these \mathcal{P}_W^i , which we call components of \mathcal{P}_W . Then define

$$C_W^\alpha(\mathcal{P}) := \sum_i C_{\mathcal{P}_W^i}^\alpha(\mathcal{P}),$$

the submodule generated by splines which vanish outside a component of \mathcal{P}_W . Then $LS^{\alpha,k}(\mathcal{P})$ is defined by

$$LS^{\alpha,k}(\mathcal{P}) := \sum_{\substack{W \in L_{\widehat{\mathcal{P}}, \widehat{\mathcal{P}}-1} \\ 0 \leq rk(W) \leq k}} C_W^\alpha(\mathcal{P}).$$

It is equivalent to let the sum in the definition of $LS^{\alpha,k}(\mathcal{P})$ run across maximal components (with respect to inclusion) occurring among lattice complexes \mathcal{P}_W with the rank of W at most k . To make this more precise, let $\Gamma_{\mathcal{P}}^k$ be the poset of components of lattice complexes \mathcal{P}_W with $rk(W) \leq k$, ordered with respect to inclusion. Let $\Gamma_{\mathcal{P}}^{k,\max}$ be the set of maximal subcomplexes appearing in $\Gamma_{\mathcal{P}}^k$. Then we have

Proposition 2.4. [11, Proposition 4.9]

$$LS^{\alpha,k}(\mathcal{P}) = \sum_{Q \in \Gamma_{\mathcal{P}}^{k,\max}} C_Q^\alpha(\mathcal{P}).$$

Since we will use this construction primarily in the cases $k = 0$ and $k = 1$, we describe $LS^{\alpha,0}(\mathcal{P})$ and $LS^{\alpha,1}(\mathcal{P})$ precisely. If γ is a face of \mathcal{P} of some dimension, we use $C_\gamma^\alpha(\mathcal{P})$ and $C_{\text{st}(\gamma)}^\alpha(\mathcal{P})$ interchangeably to denote the subring of splines which vanish outside of the star of γ , as long as no confusion results. So $C_\sigma^\alpha(\mathcal{P})$ for $\sigma \in \mathcal{P}_n$ denotes the subring of splines supported on a single facet, $C_\tau^\alpha(\mathcal{P})$ for $\tau \in \mathcal{P}_{n-1}^0$ denotes the ring of splines supported on the two facets of $\text{st}(\tau)$, etc.

Corollary 2.5. *Let $\mathcal{P} \subset \mathbb{R}^n$ be a polytopal complex. Then*

$$\begin{aligned} LS^{\alpha,0}(\mathcal{P}) &= \sum_{\sigma \in \mathcal{P}_n} C_\sigma^\alpha(\mathcal{P}) \\ LS^{\alpha,1}(\mathcal{P}) &= \sum_{\tau \in \mathcal{P}_{n-1}^0} C_\tau^\alpha(\mathcal{P}) \end{aligned}$$

Proof. For $k = 0$, the only flat $W \in L(\mathcal{P}, \mathcal{P}^{-1})$ of rank zero is the whole space \mathbb{R}^n . The corresponding lattice complex $\mathcal{P}_{\mathbb{R}^n}$ is the disjoint union of the facets of \mathcal{P} . Hence

$$LS^{\alpha,0}(\mathcal{P}) = C_{\mathbb{R}^n}^\alpha(\mathcal{P}) = \sum_{\sigma \in \mathcal{P}_n} C_\sigma^\alpha(\mathcal{P}).$$

For $k = 1$, the flats $W \in L(\mathcal{P}, \mathcal{P}^{-1})$ of rank one are precisely the hyperplanes $\text{aff}(\tau)$ with $\alpha(\tau) \geq 0$, where $\tau \in \mathcal{P}_{n-1}$. The components of the lattice complex $\mathcal{P}_{\text{aff}(\tau)}$ are the complexes $\text{st}(\gamma)$ for all γ satisfying $\text{aff}(\gamma) = \text{aff}(\tau)$. If $\gamma \in \mathcal{P}_{n-1}^0$, then $\text{st}(\gamma)$ consists of two facets and all their faces; otherwise $\gamma \in (\partial\mathcal{P})_{n-1}$ and $\text{st}(\gamma)$ consists of a single facet of \mathcal{P} and all its faces. However, as long as \mathcal{P} is hereditary and has more than one facet, every facet $\sigma \in \mathcal{P}_n$ has a codimension one face γ which is interior. Hence $\sigma \subset \text{st}(\gamma)$. It follows that $\Gamma_{\mathcal{P}}^{1,\max}$ consists of stars of interior codimension one faces of \mathcal{P} . By Proposition 2.4 we have

$$LS^{\alpha,1}(\mathcal{P}) = \sum_{\tau \in \mathcal{P}_{n-1}^0} C_\tau^\alpha(\mathcal{P})$$

□

Theorem 4.3 of [11] makes precise the sense in which $LS^{r,k}(\mathcal{P})$ is an approximation to $C^r(\mathcal{P})$. This result and its proof extend directly to mixed splines, so we state the result in this context.

Theorem 2.6. *Let $\mathcal{P} \subset \mathbb{R}^n$ be a polytopal complex. Then $LS^{\alpha,k}(\mathcal{P})$ fits into a short exact sequence*

$$0 \rightarrow LS^{\alpha,k}(\mathcal{P}) \rightarrow C^\alpha(\mathcal{P}) \rightarrow C \rightarrow 0$$

where C has codimension $\geq k+1$ and the primes in the support of C with codimension $k+1$ are contained in the set $\{I(W) | W \in L(\mathcal{P}, \mathcal{P}^{-1}) \text{ and } \text{rk}(W) = k+1\}$.

To use the submodules $LS^{\alpha,k}(\mathcal{P})$ effectively, it will be useful to fit $LS^{\alpha,k}(\mathcal{P})$ into a chain complex whose pieces are easier to understand. In the next section we describe such a complex.

3. AN INTERSECTION COMPLEX

In this section we introduce a Čech-type complex for finite sums of submodules of a given S -module M and give a criterion for its exactness. We apply this to the submodules $LS^{\alpha,k}(\mathcal{P}) \subset C^\alpha(\mathcal{P})$ in § 4.

For an integer N , let $I(k)$ be the set of all subsets of size $k \geq 1$ formed from the index set $\{1, \dots, N\}$. Thinking of $I \in I(k)$ as a k -simplex of the N -simplex Δ , we have the complex $\Delta_\bullet(S)$ with $\Delta_k(S) = \bigoplus_{I \in I(k)} S$ below whose homology is the simplicial homology of Δ with coefficients in S .

$$\Delta_\bullet(S) : S \xrightarrow{\delta_{N-1}} \bigoplus_{I \in I(N-1)} S \xrightarrow{\delta_{N-2}} \dots \xrightarrow{\delta_k} \bigoplus_{I \in I(k)} S \xrightarrow{\delta_{k-1}} \dots \xrightarrow{\delta_1} \bigoplus_{i=1}^N S \rightarrow 0$$

If $k > 0$ and $e_I \in \bigoplus_{I \in I(k)} S$ is the idempotent corresponding to $I = \{i_1, \dots, i_k\} \in I(k)$, then

$$\delta_k(e_I) = \sum_{j=1}^k (-1)^{j-1} e_{I \setminus i_j}$$

It will be convenient to augment this complex with a final map $\bigoplus_{i=1}^N S \xrightarrow{\epsilon} S$ defined by $\delta_0(e_i) = 1$ for every i . We denote this augmented complex as $\Delta_\bullet^a(S)$. The homology of $\Delta_\bullet^a(S)$ computes the *reduced* homology of Δ with coefficients in S . We extend these complexes to an S -module M by tensoring; let $\Delta_\bullet(M)$ denote $\Delta_\bullet(S) \otimes_S M$ and $\Delta_\bullet^a(M)$ denote $\Delta_\bullet^a(S) \otimes_S M$.

Now suppose $\mathcal{M} = \{M_1, \dots, M_N\}$, where each M_i is a submodule of M . For $I \subset \{1, \dots, N\}$ let M_I denote the intersection $\bigcap_{i \in I} M_i$. Define submodules $C_k(\mathcal{M}) = \bigoplus_{I \in I(k)} M_I \subset \bigoplus_{I \in I(k)} M = \Delta_k(M)$. Since $M_I \subset M_{I \setminus i}$ for every $i \in I$, the differential δ_k of $\Delta_\bullet(M)$ restricts to a map $\delta_k : C_k(\mathcal{M}) \rightarrow C_{k-1}(\mathcal{M})$, so $C_\bullet(\mathcal{M})$ is a subcomplex of $\Delta_\bullet(M)$. For example, if $N = 2$, $C_\bullet(\mathcal{M})$ is the complex

$$0 \rightarrow M_{12} \xrightarrow{\delta_1} M_1 \bigoplus M_2,$$

where $\delta_1(m) = (-m, m)$. Given any submodule $M' \subset M$ containing all the M_i , we may augment $C_\bullet(\mathcal{M})$ with the map

$$\bigoplus_{i=1}^N M_i \xrightarrow{\epsilon} M',$$

where $\epsilon(m_1, \dots, m_N) = m_1 + \dots + m_N$. We denote this augmented complex by $C_\bullet^a(\mathcal{M}, M')$.

Now consider the condition (\star) on \mathcal{M} given by

$$(\star) \quad M_I \cap (\sum_{i \in T} M_i) = \sum_{i \in T} (M_I \cap M_i) \\ \text{for every pair of subsets } I, T \subset \{1, \dots, N\}$$

We only need to check this condition on subsets I, T with $I \cap T = \emptyset$, since if there is some $j \in I \cap T$ then $M_I \subset M_j$ and both sides are equal to M_I .

Proposition 3.1. *If $\mathcal{M} = \{M_1, \dots, M_N\}$ satisfies (\star) then $H_i(C_\bullet^a(\mathcal{M}, M)) = 0$ for $i > 0$ and $H_0(C_\bullet^a(\mathcal{M}, M)) = M/(\sum_{i=1}^N M_i)$.*

Proof. The assertion $H_0(C_\bullet^a(\mathcal{M}, M)) = M/(\sum_{i=1}^N M_i)$ is always true, so we prove $H_i(C_\bullet^a(\mathcal{M}, M)) = 0$ for $i > 0$. We proceed by induction on the cardinality N of \mathcal{M} . If $N = 2$ then $C_\bullet^a(\mathcal{M}, M)$ is the complex

$$0 \rightarrow M_{12} \xrightarrow{\delta_1} M_1 \oplus M_2 \rightarrow M \rightarrow 0$$

which satisfies the conclusion of Proposition 3.1. Now suppose $N > 2$. Let $\mathcal{M}' = \{M_1, \dots, M_{N-1}\}$ and $\mathcal{N} = \{M_{1,N}, \dots, M_{N-1,N}\}$, where $M_{i,j} = M_i \cap M_j$. We have a short exact sequence of complexes $0 \rightarrow C_\bullet^a(\mathcal{M}', M) \rightarrow C_\bullet^a(\mathcal{M}, M) \rightarrow C_\bullet^a(\mathcal{N}, M_N)(-1) \rightarrow 0$, shown below. Here $C(i)$ denotes the complex C with shifted grading $C(i)_j = C_{i+j}$. This short exact sequence follows from the fact that $C_\bullet^a(\mathcal{M}, M)$ can be constructed as the mapping cone of the (appropriately signed) inclusion $C_\bullet^a(\mathcal{N}, M_N) \hookrightarrow C_\bullet^a(\mathcal{M}', M)$. It is also not difficult to check exactness of this sequence directly.

$$\begin{array}{ccccccc} & & & 0 & & 0 & 0 \\ & & & \downarrow & & \downarrow & \downarrow \\ C_\bullet^a(\mathcal{M}', M) & 0 & \longrightarrow & M_{1,\dots,N-1} \dots & \xrightarrow{\delta'_{N-1}} & \bigoplus_{i=1}^{N-1} M_i & \xrightarrow{\delta'_1} M \\ & \downarrow & & \downarrow & & \downarrow & \downarrow \\ C_\bullet^a(\mathcal{M}, M) & M_{1,\dots,N} & \xrightarrow{\delta_N} & \bigoplus_{I \in I(N-1)} M_I \dots & \xrightarrow{\delta_2} & \bigoplus_{i=1}^N M_i & \xrightarrow{\delta_1} M \\ & \downarrow & & \downarrow & & \downarrow & \downarrow \\ C_\bullet^a(\mathcal{N}, M_N)(-1) & M_{1,\dots,N} & \xrightarrow{\delta''_N} & \bigoplus_{\substack{I \in I(N-1) \\ N \in I}} M_I \dots & \xrightarrow{\delta''_2} & M_N & \xrightarrow{\delta''_1} 0 \\ & \downarrow & & \downarrow & & \downarrow & \downarrow \\ & 0 & & 0 & & 0 & 0 \end{array}$$

Clearly \mathcal{M}' satisfies (\star) , inheriting the necessary conditions from the fact that \mathcal{M} satisfies (\star) . Since $|\mathcal{M}'| = N - 1$, $H_i(C_\bullet^a(\mathcal{M}', M)) = 0$ for $i > 0$ by induction. We claim \mathcal{N} also satisfies (\star) . Interpreted for the set \mathcal{N} , the condition (\star) is

$$(\star\star) \quad M_{I \cup N} \cap (\sum_{i \in T} M_{i,N}) = \sum_{i \in T} M_{I \cup N} \cap M_i \\ \text{for every pair of subsets } I, T \subset \{1, \dots, N-1\}$$

First note that for any subset $T \subset \{1, \dots, N-1\}$, $\sum_{i \in T} M_{i,N} = M_N \cap (\sum_{i \in T} M_i)$ since \mathcal{M} satisfies (\star) . So the left hand side of $(\star\star)$ is equivalent to $M_{I \cup N} \cap (\sum_{i \in T} M_i)$. Again, since \mathcal{M} satisfies (\star) , $M_{I \cup N} \cap (\sum_{i \in T} M_i) = \sum_{i \in T} M_{I \cup N} \cap M_i$.

$$H_i(C_{\bullet}^a(\mathcal{M}, M)) = 0$$
$$0 \longrightarrow H_1(C_\bullet^a(\mathcal{M}, M)) \longrightarrow H_1(C_\bullet^a(\mathcal{N}, M_N)(-1)) \longrightarrow \\ \longrightarrow H_0(C_\bullet^a(\mathcal{M}', M)) \longrightarrow H_0(C_\bullet^a(\mathcal{M}, M)) \longrightarrow 0$$
$$\begin{aligned} H_1(C_{\bullet}^a(\mathcal{N}, M_N)(-1)) &= \frac{M_N}{\sum_{i=1}^{N-1} M_i \cap M_N} \\ &= \frac{M_N}{M_N \cap (\sum_{i=1}^{N-1} M_i)} \\ &= \frac{M_N + \sum_{i=1}^{N-1} M_i}{\sum_{i=1}^{N-1} M_i}, \end{aligned}$$
$$H_0(C_\bullet^a(\mathcal{M}', M)) = \frac{M}{\sum_{i=1}^{N-1} M_i} \rightarrow \frac{M}{\sum_{i=1}^N M_i} = H_0(C_\bullet^a(\mathcal{M}, M)).$$
☐
$$C_{\bullet}(\mathcal{M}) \rightarrow \sum_{i=1}^N M_i \rightarrow 0$$

4. INTERSECTION COMPLEX FOR SPLINES

$$LS^{\alpha,k}(\mathcal{P}) := \sum_{\mathcal{O} \in \Gamma_{\mathcal{P}}^{k,\max}} C_{\mathcal{O}}^{\alpha}(\mathcal{P}),$$

Now set $\mathcal{M}_k = \{C_{\mathcal{O}}^{\alpha}(\mathcal{P}) | \mathcal{Q} \in \Gamma_{\mathcal{P}}^{k, \max}\}$. $LS^{\alpha, k}(\mathcal{P})$ fits into the complex

$$C_\bullet(\mathcal{M}_k) \rightarrow LS^{\alpha,k}(\mathcal{P}).$$

$$C_{\mathcal{O}}^{\alpha}(\mathcal{P}) \cap C_{\mathcal{O}}^{\alpha}(\mathcal{P}) = C_{\mathcal{O} \cap \mathcal{O}}^{\alpha}(\mathcal{P}).$$

This extends to any finite intersection, hence we may write

$$C_i(\mathcal{M}_k) = \bigoplus_{\mathcal{Q}} C_{\mathcal{Q}}^{\alpha}(\mathcal{P}),$$

where \mathcal{Q} runs across all intersections of i subcomplexes from $\Gamma_{\mathcal{P}}^{k, \max}$. As will be evident below, the same subcomplex can appear multiple times as an intersection. We prove exactness of $C_{\bullet}(\mathcal{M}_1)$.

Proposition 4.1. *Let $\mathcal{M}_1 = \{C_{\tau}^{\alpha}(\mathcal{P}) | \tau \in \mathcal{P}_{n-1}^0\}$. Then the augmented complex*

$$C_{\bullet}(\mathcal{M}_1) \rightarrow LS^{\alpha,1}(\mathcal{P}) \rightarrow 0$$

is exact.

Proof. We show that \mathcal{M}_1 satisfies the condition (\star) from the previous section; then by Corollary 3.2 the proposition will be proved. First suppose given $m > 1$ codimension one faces τ_1, \dots, τ_m . If these are all faces of a common facet σ , then $\text{st}(\tau_1) \cap \dots \cap \text{st}(\tau_m) = \sigma$. Otherwise, this intersection has dimension less than n and no splines are defined on it. Hence to show (\star) for \mathcal{M}_1 amounts to showing that, given a set $T = \{\tau_1, \dots, \tau_n\}$ of codimension one faces of \mathcal{P} , the following equalities hold. Keep in mind that for two subcomplexes \mathcal{O}, \mathcal{Q} , $C_{\mathcal{O}}^{\alpha}(\mathcal{P}) \cap C_{\mathcal{Q}}^{\alpha}(\mathcal{P}) = C_{\mathcal{O} \cap \mathcal{Q}}^{\alpha}(\mathcal{P})$.

(1) For any facet $\sigma \in \mathcal{P}_n$,

$$C_{\sigma}^{\alpha}(\mathcal{P}) \cap \left(\sum_{i=1}^n C_{\text{st}(\tau_i)}^{\alpha}(\mathcal{P}) \right) = \sum_{i=1}^n C_{\sigma \cap \text{st}(\tau_i)}^{\alpha}(\mathcal{P}),$$

(2) For any codimension one face $\tau \in \mathcal{P}_{n-1}^0$,

$$C_{\text{st}(\tau)}^{\alpha}(\mathcal{P}) \cap \left(\sum_{i=1}^n C_{\text{st}(\tau_i)}^{\alpha}(\mathcal{P}) \right) = \sum_{i=1}^n C_{\text{st}(\tau) \cap \text{st}(\tau_i)}^{\alpha}(\mathcal{P})$$

(1) If $\sigma \subset \text{st}(\tau_i)$ for some $\tau_i \in T$, then both sides are equal to $C_{\sigma}^{\alpha}(\mathcal{P})$. Otherwise both sides are trivial. (2) If $\tau \in T$, then both sides are equal to $C_{\tau}^{\alpha}(\mathcal{P})$. Otherwise, set $C_T^{\alpha}(\mathcal{P}) = \sum_{i=1}^n C_{\text{st}(\tau_i)}^{\alpha}(\mathcal{P})$ and let $F \in C_{\text{st}(\tau)}^{\alpha}(\mathcal{P}) \cap C_T^{\alpha}(\mathcal{P})$. Since $\tau \notin T$, F must vanish along τ to order $\alpha(\tau)$. Letting σ_1, σ_2 be the two facets of $\text{st}(\tau)$, we see $F|_{\sigma_i} \in C_{\sigma_i}^{\alpha}(\mathcal{P})$ for $i = 1, 2$. It follows that

$$C_{\tau}^{\alpha}(\mathcal{P}) \cap C_T^{\alpha}(\mathcal{P}) = C_{\sigma_1}^{\alpha}(\mathcal{P}) \cap C_T^{\alpha}(\mathcal{P}) + C_{\sigma_2}^{\alpha}(\mathcal{P}) \cap C_T^{\alpha}(\mathcal{P}).$$

Now by (1) the intersections $C_{\sigma_i}^{\alpha}(\mathcal{P}) \cap C_T^{\alpha}(\mathcal{P})$ distribute. \square

Remark 4.2. It would be interesting to know if Proposition 4.1 holds for any \mathcal{M}_k , where $k > 1$.

Proposition 4.3. *Let $\mathcal{M}_1 = \{C_{\tau}^{\alpha}(\mathcal{P}) | \tau \in \mathcal{P}_{n-1}^0\}$, where \mathcal{P} is a pure n -dimensional hereditary polytopal complex. For a facet $\sigma \in \mathcal{P}_n$, let $\partial^0(\sigma)$ denote the set of codimension one faces of σ that are interior faces of \mathcal{P} . Set $\delta(\mathcal{P}) = \max_{\sigma \in \mathcal{P}_n} \{|\partial^0(\sigma)|\}$. The complex $C_{\bullet}(\mathcal{M}_1)$ satisfies*

$$C_k(\mathcal{M}_1) = \begin{cases} \bigoplus_{\tau \in \mathcal{P}_{n-1}^0} C_{\tau}^{\alpha}(\mathcal{P}) & \text{if } k = 1 \\ \bigoplus_{|\partial^0(\sigma)| \geq k} (C_{\sigma}^{\alpha}(\mathcal{P}))^{(|\partial^0(\sigma)| - k)} & \text{if } 2 \leq k \leq \delta(\mathcal{P}) \\ 0 & \text{if } k > \delta(\mathcal{P}) \end{cases}$$

Proof. By definition $C_1(\mathcal{M}_1)$ is the direct sum of all the submodules of \mathcal{M}_1 . In general we have

$$C_k(\mathcal{M}_1) = \bigoplus_{\mathcal{Q}} C_{\mathcal{Q}}^{\alpha}(\mathcal{P}),$$

where the direct sum runs over all subcomplexes $\mathcal{Q} \subset \mathcal{P}$ which are intersections of k distinct subcomplexes chosen from the set $\{\text{st}(\tau) | \tau \in \mathcal{P}_{n-1}^0\}$. If $k \geq 2$ then \mathcal{Q} is the intersection of two or more stars of codimension one faces, say $\text{st}(\tau_1), \text{st}(\tau_2), \dots, \text{st}(\tau_k)$. Hence \mathcal{Q} contains at most one facet, and that facet must have τ_1, \dots, τ_k as faces. So if $k \geq 2$,

$$C_k(\mathcal{M}_1) = \bigoplus_{|\partial^0(\sigma)| \geq k} (C_{\sigma}^{\alpha}(\mathcal{P}))^{(|\partial^0(\sigma)| \choose k)},$$

where $|\partial^0(\sigma)|$ is the number of edges of σ which are interior to \mathcal{P} . From this we also see that $C_k(\mathcal{M}_1) = 0$ for $k > \delta(\mathcal{P})$. \square

5. REGULARITY

In this section we briefly summarize some commutative algebra. The first chapter of [14] is an excellent introduction to the graded approach we take here. Most of the material of this section comes from this source.

Let M be a graded module over the polynomial ring $S = k[x_0, \dots, x_n]$, where k is a field of characteristic 0. Let $HF(M, d) = \dim_k M_d$ denote the Hilbert function of M in degree d . A standard result states that for $d \gg 0$, $HF(M, d)$ agrees with a polynomial function $HP(M, d)$, the Hilbert polynomial of M . As we noted in the introduction, the largest integer d for which $HF(M, d) \neq HP(M, d)$ is called the *postulation number* of M , denoted by $\wp(M)$. The degree of $HP(M, d)$ is one less than the Krull dimension of M , denoted $\dim(M)$. The *codimension* of M is defined by $\dim(S) - \dim(M) = n + 1 - \dim(M)$. M has a minimal graded free resolution

$$F_{\bullet} : 0 \rightarrow F_{\delta} \xrightarrow{\phi_r} F_{r-1} \xrightarrow{\phi_{r-1}} \dots \xrightarrow{\phi_1} F_0,$$

with $\text{coker } \phi_1 = M$. The index δ of the final free module appearing in this resolution is called the projective dimension of M , denoted $\text{pd}(M)$. For an integer a , let $S(a)$ denote the polynomial ring with grading shifted by a , so $S(a)_d \cong S_{a+d}$. An important invariant of the module M which (among other things) governs when $HF(M, d)$ becomes polynomial is the *Castelnuovo-Mumford regularity* of M .

Definition 5.1. Let M be a graded S module and $F_{\bullet} \rightarrow M$ the minimal free resolution of M , with $F_i \cong \bigoplus_j S(-a_{ij})$. The Castelnuovo-Mumford regularity of M , denoted $\text{reg}(M)$, is defined by

$$\text{reg}(M) = \max_{i,j} \{a_{i,j} - i\}.$$

Remark 5.2. Note that, according to this definition, $\text{reg}(M)$ bounds the minimal degree of generators of M as an S -module.

From Definition 5.1 one derives the following theorem. Recall an S -module M is *Cohen-Macaulay* if $\text{codim}(M) = \text{pd}(M)$.

Theorem 5.3. [14, Theorem 4.2] *Let M be a finitely generated graded module over S . Then*

- (1) $HF(M, d) = HP(M, d)$ for $d \geq \text{reg}(M) + \text{pd}(M) - n$. Equivalently, $\wp(M) \leq \text{reg}(M) + \text{pd}(M) - n - 1$.
- (2) If M is a Cohen-Macaulay module, the bound in (1) is sharp.

Another characterization of regularity is obtained via local cohomology, so we introduce this notion. See [14, Appendix 1] for more details. Let Q be an ideal of S . The local cohomology modules $H_Q^i(M)$ of M with respect to Q are the right derived functors of the the Q -torsion functor $H_Q^0(_)$, where

$$H_Q^0(M) = \{x \in M \mid Q^j x = 0 \text{ for some } j \geq 0\}.$$

We will only be concerned with the case $Q = m$, where $m = (x_0, \dots, x_n)$ is the graded maximal ideal of S .

Theorem 5.4 (Theorem 4.3 of [14]). *Let $m \subset S$ be the maximal ideal of S and M a graded S -module. Then*

$$\text{reg}(M) = \max_i (\max_e \{e \mid H_m^i(M)_e \neq 0\} + i)$$

The benefit of this description of regularity is that it interacts well with short exact sequences. For instance, the following result is a straightforward application of Theorem 5.4.

Proposition 5.5. [13, Corollary 20.19] *Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be a graded exact sequence of finitely generated S modules. Then*

- (1) $\text{reg}(A) \leq \max\{\text{reg}(B), \text{reg}(C) + 1\}$
- (2) $\text{reg}(B) \leq \max\{\text{reg}(A), \text{reg}(C)\}$
- (3) $\text{reg}(C) \leq \max\{\text{reg}(A) - 1, \text{reg}(B)\}$

Proposition 5.5 can be extended to bound the regularity of a module appearing in an exact sequence of any length by breaking the exact sequence into short exact pieces. We will use the following corollary to Proposition 5.5.

Corollary 5.6. *Let $m \geq 0$ and*

$$0 \rightarrow C_m \rightarrow C_{m-1} \rightarrow \dots \rightarrow C_0 \rightarrow M \rightarrow 0$$

an exact sequence of S -modules. Then

$$\text{reg}(M) \leq \max_i \{\text{reg}(C_i) - i\}$$

One more concept that is relevant to our situation is that of depth. The *depth* of a graded S -module M with respect to the homogeneous maximal ideal m , denoted $\text{depth}(M)$, is the length of a maximal sequence $\{f_1, \dots, f_k\} \subset m$ satisfying that f_1 is a non-zerodivisor on M and f_l is a non-zerodivisor on $M/(\sum_{i=1}^{l-1} f_i M)$ for $l = 2, \dots, k$. Such a sequence is called an M -sequence. We will use the following result of Auslander and Buchsbaum to move back and forth between the notions of depth and projective dimension.

Theorem 5.7 (Auslander-Buchsbaum). *Let M be an $S = k[x_0, \dots, x_n]$ -module. Then*

$$\text{depth}(M) + \text{pd}(M) = n + 1.$$

Observe that, according to this formula, $\text{pd}(M) \leq n + 1$. This inequality is known as the Hilbert syzygy theorem.

The following proposition is one of the ingredients used in the proof of the Gruson-Lazarsfeld-Peskine theorem on bounding the regularity of curves in projective space [14, Proposition 5.5]. It is the main tool we will use for bounding regularity of spline modules.

Proposition 5.8. *Let M be an S -module and $N \subset M$ a submodule of M with $\dim(M/N) < \text{depth}(M)$, or equivalently $\text{codim}(M/N) > \text{pd}(M)$. Then $\text{reg}(M) \leq \text{reg}(N)$.*

Proof. We prove $\text{reg}(M) \leq \text{reg}(N)$ if $\dim(M/N) < \text{depth}(M)$. The equivalence of the statements $\dim(M/N) < \text{depth}(M)$ and $\text{codim}(M/N) > \text{pd}(M)$ follows directly from Theorem 5.7. Set $d = \text{depth}(M)$. By [14, Proposition A1.16], $H_m^i(M) = 0$ for $i < d$ and $H_m^i(M/N) = 0$ for $i > \dim(M/N)$. The long exact sequence in local cohomology resulting from the short exact sequence

$$0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$$

yields a surjection $H_m^d(N) \twoheadrightarrow H_m^d(M)$ and isomorphisms $H_m^i(N) \cong H_m^i(M)$ for $i > d$. Since $H_m^i(M) = 0$ for $i < d$, Theorem 5.4 yields $\text{reg}(N) \geq \text{reg}(M)$. \square

5.1. High degree generators for splines. We give a construction motivating the regularity bounds we derive in Corollary 6.3, Theorem 6.7, and Theorem 7.2. These results suggest that in general regularity bounds for $C^\alpha(\mathcal{P})$ might be obtained by taking the maximal sum of smoothness parameters $\alpha(\tau) + 1$ appearing in certain subcomplexes of \mathcal{P} . In the following example, starting with a polytope $\sigma \subset \mathbb{R}^n$, we construct a polytopal complex \mathcal{P} so that $\sigma \in \mathcal{P}_n$ and $C^\alpha(\hat{\mathcal{P}})$ ($C^\alpha(\mathcal{P})$ if \mathcal{P} is central) has a minimal generator supported the facet $\hat{\sigma}$ (σ if \mathcal{P} is central). Such generators have degree $\sum_{\tau \in \sigma_{n-1}} \alpha(\tau) + 1$. Since $\text{reg}(C^\alpha(\mathcal{P}))$ in particular bounds the degrees of generators of $C^\alpha(\mathcal{P})$ (see Remark 5.2), this construction indicates that a bound on $\text{reg}(C^\alpha(\mathcal{P}))$ will need to be at least as large as the maximal sum of smoothness parameters over codimension one faces occurring in any facet of \mathcal{P} (or at least boundary facets - see Conjecture 9.1). This example generalizes the construction in [11, Theorem 5.7].

For simplicity we restrict the construction to the case of uniform smoothness without imposing boundary vanishing. The generalization to arbitrary smoothness parameters should be clear.

Example 5.9. Suppose that $A \subset \mathbb{R}^n$ is a polytope with a codimension one face $\tau \in A_{n-1}$ so that $\partial A \setminus \tau$ is the graph of a piecewise linear function over τ . Remark 5.10 below shows that this can be accomplished for any polytope by a projective change of coordinates.

For instance this is true if A is the join of τ with the origin $\mathbf{0} \in \mathbb{R}^n$. Let l_τ be a choice of affine form vanishing on τ and let x_1, \dots, x_n be coordinates on \mathbb{R}^n . We further assume that

- (1) τ is parallel to the coordinate hyperplane $x_n = 0$
- (2) A lies between the hyperplanes $x_n = 0$ and $l_\tau = 0$.
- (3) For any two codimension one faces $\gamma_1, \gamma_2 \in A_{n-1} \setminus \tau$, $\text{aff} \gamma_1$ and $\text{aff}(\gamma_2)$ intersect the coordinate hyperplane $x_n = 0$ in distinct linear subspaces of codimension 2.

(1) can be obtained by rotating the original polytope, (2) and (3) can be obtained by translation. If A is the join of τ with the origin, (3) may be obtained by slight perturbations of the non-zero vertices of A (within the plane $l_\tau = 0$).

Let B be the reflection of A across the hyperplane $x_n = 0$. For a face $\gamma \in A$, let $\bar{\gamma}$ denote the corresponding face of B obtained by reflection. For $\gamma \in A_{n-1} \setminus \tau$, let $\sigma(\gamma)$ denote the polytope formed by taking the convex hull of γ and $\bar{\gamma}$. Now define $\mathcal{P}(A)$ as the polytopal complex with facets A, B and $\{\sigma(\gamma) \mid \gamma \neq \tau \in A_{n-1}\}$. See Figure 3 for examples of this construction in \mathbb{R}^2 and \mathbb{R}^3 . Take the cone $\widehat{\mathcal{P}(A)} \subset \mathbb{R}^n$

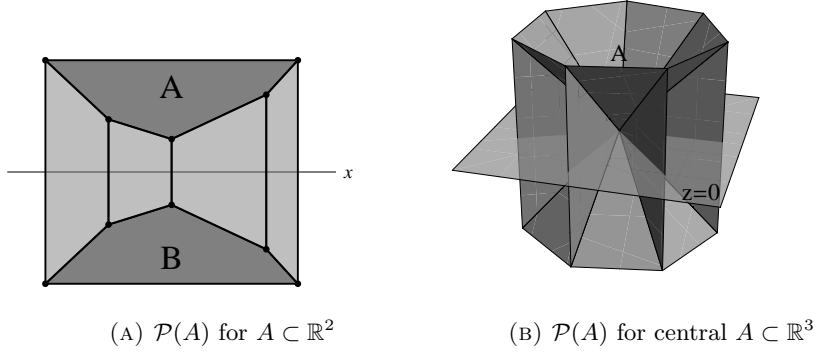


FIGURE 3

over $\mathcal{P}(A)$ and consider the graded $S = \mathbb{R}[x_0, \dots, x_n]$ -module $C^r(\widehat{\mathcal{P}(A)})$. Let $\phi_B : C^r(\widehat{\mathcal{P}(A)}) \rightarrow S$ be the S -linear map obtained by restricting splines $F \in C^r(\widehat{\mathcal{P}(A)})$ to the facet \widehat{B} . This is a splitting of the inclusion $S \rightarrow C^r(\widehat{\mathcal{P}(A)})$ as global polynomials on $\widehat{\mathcal{P}(A)}$. Let $NT^r(\widehat{\mathcal{P}(A)})$ be the kernel of ϕ_B . Then

$$C^r(\widehat{\mathcal{P}(A)}) \cong S \oplus NT^r(\widehat{\mathcal{P}(A)}).$$

Let $S' = \mathbb{R}[x_0, \dots, x_{n-1}]$ and, for $f \in S$, set $\bar{f} = f(x_0, \dots, x_{n-1}, 0)$. Define an S -linear map $\phi : C^r(\widehat{\mathcal{P}(A)}) \cong S \oplus NT^r(\widehat{\mathcal{P}(A)}) \rightarrow S'$ by

$$(f, F) \rightarrow \bar{F}_{\widehat{A}},$$

where $f \in S$, $F \in NT^r(\widehat{\mathcal{P}(A)})$, and $F_{\widehat{A}}$ is the restriction of F to the facet \widehat{A} . Set $\Lambda(A) = \prod_{\gamma \neq \tau \in A_{n-1}} L_\gamma^{r+1}$, where $L_\gamma = l_{\bar{\gamma}}$ is a choice of homogeneous form vanishing on $\bar{\gamma}$. We claim that the image of ϕ is the principal ideal

$$I = \langle \overline{\Lambda(A)} \rangle.$$

ϕ is surjective since the spline $G(A)$, defined by

$$G(A)_\sigma = \begin{cases} 0 & \sigma \neq A \\ \Lambda(A) & \sigma = A, \end{cases}$$

goes to the generator of I under ϕ . To see that $\text{im}(\phi) \subset I$, let $F \in NT^r(\widehat{\mathcal{P}(A)})$. Then, since $F_{\widehat{B}} = 0$, $L_{\bar{\gamma}}^{r+1} | F_{\sigma(\gamma)}$ for every $\bar{\gamma} \neq \bar{\tau} \in B_{n-1}$. We also have $L_\gamma^{r+1} | (F_{\widehat{A}} - F_{\sigma(\gamma)})$ for every $\gamma \neq \tau \in A_{n-1}$. Hence $F_{\widehat{A}} \in \cap_{\gamma \neq \tau \in A_{n-1}} \langle L_\gamma^{r+1}, L_{\bar{\gamma}}^{r+1} \rangle$. But L_γ and

$L_{\bar{\gamma}}$ differ at most by a scalar multiple and a sign on the variable x_n , so $\overline{L_{\gamma}} = \overline{L_{\bar{\gamma}}}$ and

$$\phi(F) \in \bigcap_{\gamma \neq \tau \in A_{n-1}} \langle \overline{L_{\gamma}^{r+1}} \rangle = \langle \prod_{\gamma \neq \tau \in A_{n-1}} \overline{L_{\gamma}^{r+1}} \rangle = \langle \overline{\Lambda(A)} \rangle$$

as claimed. Property (3) above is used in the first equality - this guarantees all the forms $\overline{L_{\gamma}}$ are distinct. It follows that the spline $G(A)$, which is supported only on the facet \hat{A} and generates splines supported on \hat{A} , is a minimal generator of $C^r(\widehat{\mathcal{P}(A)})$.

If A is the join of τ with $\mathbf{0}$, then $\mathcal{P}(A)$ is central and $C^r(\mathcal{P}(A))$ is graded over the polynomial ring $R = \mathbb{R}[x_1, \dots, x_n]$. In this case it is unnecessary to take the cone over $\mathcal{P}(A)$ above.

Remark 5.10. Given a convex polytope $A \subset \mathbb{R}^n \subset \mathbb{P}_{\mathbb{R}}^n$ and a choice τ of codimension one face, there is a projective change of coordinates which makes $\partial A \setminus \tau$ into the graph of a piecewise linear function over τ . If A is the join of τ with the origin $\mathbf{0} \in \mathbb{R}^n$, then this is easily done by a linear transformation. Otherwise, this can be accomplished by choosing a hyperplane $H \subset \mathbb{R}^n$ which is parallel to τ and very close to P without intersecting P . Then make a projective change of coordinates which sends H to the hyperplane at infinity (this argument is due to Sergei Ivanov). As long as H is chosen close enough to τ , this has the effect of making the face τ huge and the rest of the polytope the graph of a piecewise linear function over τ (once we restrict to affine coordinates again). Hence, given any polytope $A \subset \mathbb{R}^n$ and a choice of codimension one face $\tau \in A_{n-1}$, the construction in Example 5.9 allows us to build a polytopal complex $\mathcal{P}(A)$ so that $\partial^0 A = \partial A \setminus \tau$ and the generator of $C_{\hat{A}}^r(\widehat{\mathcal{P}(A)})$ is a minimal generator of $C^r(\widehat{\mathcal{P}(A)})$.

Remark 5.11. The construction in Example 5.9 is inherently nonsimplicial. Some other construction needs to be used to obtain high degree generators in the simplicial case. In the planar simplicial case, there is an example in [28] of a planar simplicial complex Δ with minimal generator in degree $2r + 2$.

6. BOUNDING REGULARITY FOR LOW PROJECTIVE DIMENSION

In this section we combine the observations so far to bound the regularity of the spline algebra $C^\alpha(\mathcal{P})$, where $\mathcal{P} \subset \mathbb{R}^{n+1}$ is a central, pure, hereditary, $(n + 1)$ -dimensional polytopal complex. Recall a central complex is one in which the intersection of all interior codimension one faces is nonempty. We assume this intersection contains the origin and that $\alpha(\tau) = -1$ for every codimension one face $\tau \in \mathcal{P}_n$ so that $\mathbf{0} \notin \text{aff}(\tau)$; this makes the ring $C^\alpha(\mathcal{P})$ a graded $S = \mathbb{R}[x_0, \dots, x_n]$ -algebra with respect to the standard grading on S . The following corollary is critical to our analysis.

Corollary 6.1. [8, Proposition 3.4] *If \mathcal{P} is a central, pure, hereditary, $(n + 1)$ -dimensional polytopal complex, then*

- (1) $\text{pd}(C^\alpha(\mathcal{P})) \leq n - 1$
- (2) $\varphi(C^\alpha(\mathcal{P})) \leq \text{reg}(C^\alpha(\mathcal{P})) - 2$.

Proof. (1) follows from Lemma 2.1. $C^\alpha(\mathcal{P})$ is the kernel of a map between free S -modules, so it is a second syzygy module. By the Hilbert syzygy theorem, any S -module has projective dimension at most $n + 1$. Since $C^\alpha(\mathcal{P})$ is a second syzygy module, $\text{pd}(C^\alpha(\mathcal{P})) \leq n - 1$. (2) follows from (1) and Theorem 5.3. \square

Theorem 6.2. *Let $\mathcal{P} \subset \mathbb{R}^{n+1}$ be a pure $(n+1)$ -dimensional hereditary polytopal complex which is central. Then*

$$\operatorname{reg}(C^\alpha(\mathcal{P})) \leq \operatorname{reg}(LS^{\alpha, n-1}(\mathcal{P}))$$

More generally, if $\operatorname{pd}(C^\alpha(\mathcal{P})) \leq k$, then

$$\operatorname{reg}(C^\alpha(\mathcal{P})) \leq \operatorname{reg}(LS^{\alpha, k}(\mathcal{P}))$$

Proof. The first statement follows from the second by Corollary 6.1. To prove the second statement, note that by Theorem 2.6, the cokernel of the inclusion $LS^{\alpha, k}(\mathcal{P})$ has codimension at least $k+1$. By Proposition 5.8, $\operatorname{reg}(C^\alpha(\mathcal{P})) \leq \operatorname{reg}(LS^{\alpha, k}(\mathcal{P}))$. \square

To simplify the statements of later results, we introduce some additional notation. Given a pure $(n+1)$ -dimensional subcomplex $\mathcal{Q} \subset \mathcal{P}$, let $\partial(\mathcal{Q})$ denote the set of n dimensional boundary faces of \mathcal{Q} . Define

$$\Lambda(\mathcal{Q}) = \prod_{\gamma \in (\partial(\mathcal{Q}))_n} l_\gamma^{\alpha(\gamma)+1}$$

and set

$$\lambda(\mathcal{Q}) = \deg(\Lambda(\mathcal{Q})) = \sum_{\gamma \in (\partial(\mathcal{Q}))_n} (\alpha(\gamma) + 1).$$

As a first application of Theorem 6.2, we give a bound on the degree of generators of $C^\alpha(\mathcal{P})$ when $C^\alpha(\mathcal{P})$ is free.

Corollary 6.3. *Suppose $C^\alpha(\mathcal{P})$ is free and set $f(\mathcal{P}) = \max\{\lambda(\sigma) | \sigma \in \mathcal{P}_{n+1}\}$. Then $C^\alpha(\mathcal{P})$ is generated in degrees at most $f(\mathcal{P})$.*

Proof. For a free module, regularity is the maximum degree of generators (this follows from Definition 5.1), so we need to show $\operatorname{reg}(C^\alpha(\mathcal{P})) \leq f(\mathcal{P})$. $C^\alpha(\mathcal{P})$ is free iff $\operatorname{pd}(C^\alpha(\mathcal{P})) = 0$. By Theorem 6.2,

$$\operatorname{reg}(C^\alpha(\mathcal{P})) \leq \operatorname{reg}(LS^{\alpha, 0}(\mathcal{P})).$$

By Corollary 2.5, $LS^{\alpha, 0} = \sum_{\sigma \in \mathcal{P}_{n+1}} C_\sigma^\alpha(\mathcal{P})$. Since the support of each summand is disjoint, this is a direct sum, so $\operatorname{reg}(LS^{\alpha, 0}(\mathcal{P})) = \max\{\operatorname{reg}(C_\sigma^\alpha(\mathcal{P})) | \sigma \in \mathcal{P}_{n+1}\}$. Also, $C_\sigma^\alpha(\mathcal{P})$ consists of splines F supported on the single facet σ . Such splines are characterized by $F|_\sigma$ being a polynomial multiple of $\Lambda(\sigma)$. It follows that $C_\sigma^\alpha(\mathcal{P}) \cong S(-\lambda(\sigma))$. Hence

$$\operatorname{reg}(LS^{\alpha, 0}(\mathcal{P})) = \max\{\lambda(\sigma) | \sigma \in \mathcal{P}_{n+1}\} = f(\mathcal{P}).$$

\square

We now apply Theorem 6.2 to the case where $C^\alpha(\mathcal{P})$ has projective dimension at most one. In particular, this includes central complexes in \mathbb{R}^3 by Corollary 6.1.

Theorem 6.4. *Suppose $\operatorname{pd}(C^\alpha(\mathcal{P})) \leq 1$. Let $f(\mathcal{P}) = \max\{\lambda(\sigma) | \sigma \in \mathcal{P}_{n+1}\}$ and $T = \max_{\tau \in \mathcal{P}_n^0} \{\operatorname{reg}(C_\tau^\alpha(\mathcal{P}))\}$. Then $\operatorname{reg}(C^\alpha(\mathcal{P})) \leq \max\{f(\mathcal{P}) - 1, T\}$.*

Proof. By Corollary 6.2,

$$\operatorname{reg}(C^\alpha(\mathcal{P})) \leq \operatorname{reg}(LS^{\alpha, 1}(\mathcal{P})).$$

By Proposition 4.1, $LS^{\alpha, 1}(\mathcal{P})$ fits into the exact sequence

$$C_\bullet(\mathcal{M}_1) \rightarrow LS^{\alpha, 1}(\mathcal{P}) \rightarrow 0.$$

From Proposition 4.3,

$$C_k(\mathcal{M}_1) = \begin{cases} \bigoplus_{\tau \in \mathcal{P}_n^0} C_\tau^\alpha(\mathcal{P}) & \text{if } k = 1 \\ \bigoplus_{|\partial^0(\sigma)| \geq k} (C_\sigma^\alpha(\mathcal{P}))^{(|\partial^0(\sigma)| \choose k)} & \text{if } 2 \leq k \leq \delta(\mathcal{P}) \\ 0 & \text{if } k > \delta(\mathcal{P}) \end{cases},$$

where $\delta(\mathcal{P}) = \max_{\sigma \in \mathcal{P}_{n+1}} \{|\partial^0(\sigma)|\}$. As we saw in the proof of Corollary 6.3, $C_\sigma^\alpha(\mathcal{P}) \cong S(-\lambda(\sigma))$, hence

$$\text{reg}(C_k(\mathcal{M}_1)) = \max\{\lambda(\sigma) \mid \sigma \in \mathcal{P}_{n+1}\} \leq f(\mathcal{P})$$

for every k with $2 \leq k \leq \delta(\mathcal{P})$. Now the conclusion follows from Corollary 5.6. \square

At this point we see that to obtain more precise results for projective dimension one it is necessary to understand the ring $C_\tau^\alpha(\mathcal{P})$ of splines vanishing outside the star of a codimension one face.

Proposition 6.5. *Let $\tau \in \mathcal{P}_n^0$ be an interior codimension one face of \mathcal{P} , and σ_1, σ_2 the two facets of $\text{st}(\tau)$, the star of τ . Set $L_\tau = l_\tau^{\alpha(\tau)+1}$, $L_1 = \Lambda(\sigma_1)/L_\tau$, $L_2 = \Lambda(\sigma_2)/L_\tau$. Define the ideal $K(\tau)$ by*

$$K(\tau) = \langle L_1, L_2, L_\tau \rangle$$

We have a graded isomorphism

$$C_\tau^\alpha(\mathcal{P}) \cong \begin{cases} S(-\deg L_\tau - \deg L_2) \oplus S(-\deg L_1) & \text{if } L_1 \in \langle L_2, L_\tau \rangle \\ S(-\deg L_\tau - \deg L_1) \oplus S(-\deg L_2) & \text{if } L_2 \in \langle L_1, L_\tau \rangle \\ S(-\deg L_1 - \deg L_2) \oplus S(-\deg L_\tau) & \text{if } L_\tau \in \langle L_1, L_2 \rangle \\ \text{syz}(K(\tau)) & \text{otherwise,} \end{cases}$$

where $\text{syz}(K(\tau))$ is the module of syzygies on the ideal $K(\tau)$.

Proof. Let $F \in C_\tau^\alpha(\mathcal{P})$ and set $F_1 = F|_{\sigma_1}$, $F_2 = F|_{\sigma_2}$. Then there are polynomials G_1, G_2, G_3 satisfying the following relations.

$$\begin{aligned} F_1 &= G_1 L_1 \\ F_2 &= G_2 L_2 \\ F_2 - F_1 &= G_3 L_\tau \end{aligned}$$

Taking the alternating sum of the above equations yields

$$(3) \quad G_1 L_1 - G_2 L_2 + G_3 L_\tau = 0.$$

Hence $F = (F_1, F_2)$ gives rise to a syzygy on the columns of the matrix

$$M = \begin{bmatrix} L_1 & L_2 & L_\tau \end{bmatrix}$$

Now suppose given a syzygy (G_1, G_2, G_3) on the columns of M . We obtain a spline $F \in C_\tau^\alpha(\mathcal{P})$ by setting $F_1 = G_1 L_1$, $F_2 = G_2 L_2$, hence $C_\tau^\alpha(\mathcal{P})$ is isomorphic to the syzygies on the columns M . If $K(\tau)$ is minimally generated by L_1, L_2 , and L_τ , we obtain $C_\tau^\alpha(\mathcal{P}) \cong \text{syz}(K(\tau))$. Otherwise we obtain the cases listed above. For instance, if $L_1 \in \langle L_2, L_\tau \rangle$, then there exist polynomials $f, g \in S$ so that $L_1 = f L_2 + g L_\tau$ and $\text{syz}(M)$ is generated by

$$\begin{bmatrix} 0 \\ L_\tau \\ -L_2 \end{bmatrix}, \begin{bmatrix} 1 \\ -f \\ -g \end{bmatrix},$$

of degrees $\deg L_2 + \deg L_\tau$ and $\deg L_1$, respectively. The other cases follow similarly. \square

Proposition 6.6. *Let $\mathcal{P} \subset \mathbb{R}^3$ be a central complex, and $\tau \in \mathcal{P}_2^0$ a codimension one face of \mathcal{P} . Define*

$$\lambda(\tau) = \lambda(\text{st}(\tau)) + \alpha(\tau) + 1 = \sum_{\gamma \in (\text{st}(\tau))_2} \alpha(\gamma) + 1.$$

Then $\text{reg}(C_\tau^\alpha(\mathcal{P})) \leq \lambda(\tau) - 1$ unless $\alpha(\gamma) = -1$ for all $\gamma \neq \tau \in (\text{st}(\tau))$, when $\text{reg}(C_\tau^\alpha(\mathcal{P})) = \alpha(\tau) + 1$.

Proof. Let L_1, L_2, L_τ be as defined in proposition 6.5. Then

$$\begin{aligned} \deg L_1 &= \left(\sum_{\gamma \in (\sigma_1)_2} (\alpha(\gamma) + 1) \right) - \alpha(\tau) - 1 \\ \deg L_2 &= \left(\sum_{\gamma \in (\sigma_2)_2} (\alpha(\gamma) + 1) \right) - \alpha(\tau) - 1 \\ \deg L_\tau &= \alpha(\tau) + 1, \end{aligned}$$

If the ideal $K(\tau) = \langle L_1, L_2, L_\tau \rangle$ is not minimally generated by L_1, L_2 , and L_τ , then $C_\tau^\alpha(\mathcal{P})$ is free, generated in degrees indicated by Proposition 6.5. By that description $\text{reg}(C_\tau^\alpha(\mathcal{P})) \leq \lambda(\tau) - 1$ unless $\alpha(\gamma) = -1$ for all $\gamma \neq \tau \in (\text{st}(\tau))$, when $\text{reg}(C_\tau^\alpha(\mathcal{P})) = \alpha(\tau) + 1$. So assume $K(\tau)$ is minimally generated by L_1, L_2, L_τ and $C_\tau^\alpha(\mathcal{P}) \cong \text{syz}(K(\tau))$.

We define a submodule $N(\tau)$ of $C_\tau^\alpha(\mathcal{P})$ as follows. Let σ_1, σ_2 be the two facets of $\text{st}(\tau)$ and $Se_1 + Se_2$ the free S -module on generators e_1, e_2 corresponding to σ_1, σ_2 . Define $N(\tau)$ to be the submodule of $C_\tau^\alpha(\mathcal{P})$ generated by $F_1 = \Lambda(\sigma_1)e_1, F_2 = \Lambda(\sigma_2)e_2$, and $F_\tau = \Lambda(\text{st}(\tau))(e_1 + e_2)$. There is a single nontrivial syzygy among F_1, F_2, F_τ given by $L_\tau F_\tau = L_2 F_1 + L_1 F_2$. So $N(\tau)$ has minimal free resolution

$$\begin{array}{c} S(-\lambda(\sigma_1)) \\ \oplus \\ 0 \longrightarrow S(-\lambda(\text{st}(\tau)) - \alpha(\tau) - 1) \longrightarrow S(-\lambda(\text{st}(\tau))) \\ \oplus \\ S(-\lambda(\sigma_2)) \end{array}$$

From Definition 5.1 and the free resolution above we see that $\text{reg}(N(\tau)) = \lambda(\text{st}(\tau)) + \alpha(\tau) = \lambda(\tau) - 1$.

Now we show $\text{codim}(C_\tau^\alpha(\mathcal{P})/N(\tau)) \geq 2$. It suffices to show that $(C_\tau^\alpha(\mathcal{P}))_P = N(\tau)_P$ for every prime of codimension one. Since S is a UFD, primes of codimension one are principle, generated by a single irreducible polynomial. If $P \neq \langle l_\gamma \rangle$ for any $\gamma \in (\text{st}(\tau))_2$ then

$$(C_\tau^\alpha(\mathcal{P}))_P = N(\tau)_P = S_P^2.$$

If $P = \langle l_\gamma \rangle$ for some $\gamma \in \partial^0(\text{st}(\tau))$, then

$$(C_\tau^\alpha(\mathcal{P}))_P = N(\tau)_P = l_\gamma^{\alpha(\gamma)+1} S_P \oplus S_P.$$

if $\text{aff}(\gamma)$ meets only one face $\gamma \in (\text{st}(\tau))_2$ or

$$(C_\tau^\alpha(\mathcal{P}))_P = N(\tau)_P = l_\gamma^{\alpha(\gamma)+1} S_P \oplus l_\gamma^{\alpha(\gamma)+1} S_P$$

If $\text{aff}(\gamma)$ meets both σ_1 and σ_2 in a codimension one face. If $P = \langle l_\tau \rangle$, then

$$(C_\tau^\alpha(\mathcal{P}))_P = N(\tau)_P = (C^{\alpha(\tau)}(\text{st}(\tau)))_P.$$

$\text{pd}(C_\tau^\alpha(\mathcal{P})) \leq 1$ follows by Corollary 6.1, because we assumed $\mathcal{P} \subset \mathbb{R}^3$. Since $\text{codim}(C_\tau^\alpha(\mathcal{P})/N(\tau)) \geq 2$,

$$\text{reg}(C_\tau^\alpha(\mathcal{P})) \leq \text{reg}(N(\tau)) = \lambda(\tau) - 1$$

follows from Proposition 5.8. \square

Theorem 6.7. *Let $\mathcal{P} \subset \mathbb{R}^3$ be a pure 3-dimensional polytopal complex which is central and set $e(\mathcal{P}) = \max\{\lambda(\tau) | \tau \in \mathcal{P}_2^0\}$. Then*

- (1) $\text{reg}(C^\alpha(\mathcal{P})) \leq e(\mathcal{P}) - 1$
- (2) $\wp(C^\alpha(\mathcal{P})) \leq e(\mathcal{P}) - 3$

In particular, $HP(C^\alpha(\mathcal{P}), d) = \dim_{\mathbb{R}} C_d^r(\mathcal{P})$ for $d \geq e(\mathcal{P}) - 2$.

Proof. (1) follows by applying Theorem 6.4 to Proposition 6.6. (2) follows from (1) by Corollary 6.1. \square

Example 1.1 indicates that the bound given in Theorem 6.7 can be far from optimal. In the next section we bound $\text{reg}(C_\tau^\alpha(\Delta))$ more precisely for $\Delta \subset \mathbb{R}^3$ a central simplicial complex.

7. SIMPLICIAL REGULARITY BOUND

In this section we analyze the regularity of the ring of splines $C_\tau^\alpha(\Delta)$ vanishing outside the star of 2-face, for $\Delta \subset \mathbb{R}^3$ a pure three-dimensional hereditary simplicial complex which is central. Again we assume $\alpha(\tau) = -1$ for $\tau \in \Delta_2$ with $\mathbf{0} \notin \text{aff}(\tau)$, so that $C^\alpha(\Delta)$ is a graded module over the polynomial ring $S = \mathbb{R}[x, y, z]$. This means that $\text{st}(\tau)$ has at most five 2-faces γ for which $\alpha(\gamma) \geq 0$ ($\alpha(\tau) \geq 0$ is required). We prove the following theorem.

Theorem 7.1. *Let $\tau \in \Delta_2^0$ be a 2-face. Define*

$$M(\tau) = (\alpha(\tau) + 1) + \max\{(\alpha(\gamma_1) + 1) + (\alpha(\gamma_2) + 1) | \gamma_1 \neq \gamma_2 \in (\text{st}(\tau))_2\}.$$

Then $\text{reg}(C_\tau^\alpha(\Delta)) \leq M(\tau)$.

Before proving Theorem 7.1 we derive a couple of corollaries.

Theorem 7.2. *Let $\Delta \subset \mathbb{R}^3$ be a pure 3-dimensional hereditary simplicial complex which is central. For $\tau \in \Delta_2^0$, let $M(\tau)$ be defined as in Theorem 7.1. Then*

- (1) $\text{reg}(C^\alpha(\Delta)) \leq \max\{M(\tau) | \tau \in \Delta_2^0\}$
- (2) $\wp(C^\alpha(\Delta)) \leq \max\{M(\tau) | \tau \in \Delta_2^0\} - 2$

In particular, $HP(C^\alpha(\Delta), d) = \dim_{\mathbb{R}} C^r(\Delta)_d$ for $d \geq \max\{M(\tau) | \tau \in \Delta_2^0\} - 1$.

Proof. (1) follows by applying Theorem 6.4 to Theorem 7.1, (2) follows by applying Theorem 5.3 to (1). \square

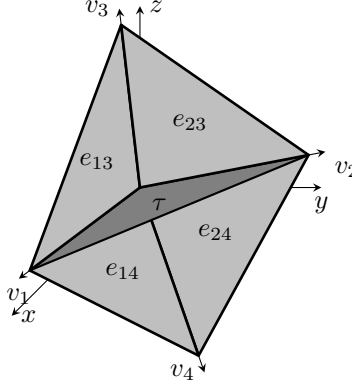
Setting $\alpha(\tau) = r$ for all $\tau \in \Delta_2^0$, we obtain

Corollary 7.3. *Let $\Delta \subset \mathbb{R}^3$ be a pure 3-dimensional hereditary simplicial complex which is central. Then*

- (1) $\text{reg}(C^\alpha(\Delta)) \leq 3r + 3$
- (2) $\wp(C^\alpha(\Delta)) \leq 3r + 1$

In particular, $HP(C^r(\Delta), d) = \dim_{\mathbb{R}} C^r(\Delta)_d$ for $d \geq 3r + 2$.

This result was obtained in the case of $C^r(\widehat{\Delta})$, for simplicial $\Delta \subset \mathbb{R}^2$, by Hong [18] and Ibrahim and Schumaker [19] (see Table 1 in the introduction). Before proving Theorem 7.2 we set up some notation. Figure 4 depicts our situation. We will abuse notation and write v_i both for the corresponding edge of $\text{st}(\tau)$ and for the vector we obtain by taking positive real multiples of this edge.

FIGURE 4. $\text{st}(\tau)$

Let $u_1, u_2 \in S$ be the forms corresponding to the 2-faces e_{13}, e_{23} , let w_1, w_2 be the forms corresponding to the 2-faces e_{14}, e_{24} , and l_τ be the form corresponding to τ (for now do this without coordinates). Let $\alpha_\tau = \alpha(\tau) + 1, \alpha_1 = \alpha(e_{13}) + 1, \alpha_2 = \alpha(e_{23}) + 1, \beta_1 = \alpha(e_{14}) + 1, \beta_2 = \alpha(e_{24}) + 1$ be the exponents to appear on $l_\tau, u_1, u_2, w_1, w_2$ corresponding to the smoothness parameters specified by α . The following lemma is a special case of Proposition 6.5.

Lemma 7.4. *Let $K(\tau) = \langle l_\tau^{\alpha_\tau}, u_1^{\alpha_1} u_2^{\alpha_2}, w_1^{\beta_1} w_2^{\beta_2} \rangle$. Then we have a graded isomorphism*

$$C_\tau^\alpha(\Delta) \cong \begin{cases} S(-\alpha_\tau - \beta_1 - \beta_2) \oplus S(-\alpha_1 - \alpha_2) & \text{if } u_1^{\alpha_1} u_2^{\alpha_2} \in \langle w_1^{\beta_1} w_2^{\beta_2}, l_\tau^{\alpha_\tau} \rangle \\ S(-\alpha_\tau - \alpha_1 - \alpha_2) \oplus S(-\beta_1 - \beta_2) & \text{if } w_1^{\beta_1} w_2^{\beta_2} \in \langle u_1^{\alpha_1} u_2^{\alpha_2}, l_\tau^{\alpha_\tau} \rangle \\ S(-\alpha_1 - \alpha_2 - \beta_1 - \beta_2) \oplus S(-\alpha_\tau) & \text{if } l_\tau^{\alpha_\tau} \in \langle u_1^{\alpha_1} u_2^{\alpha_2}, w_1^{\beta_1} w_2^{\beta_2} \rangle \\ \text{syz}(K(\tau)) & \text{otherwise,} \end{cases}$$

where $\text{syz}(K(\tau))$ is the module of syzygies on the ideal $K(\tau)$.

Proof of Theorem 7.1. If $u_1^{\alpha_1} u_2^{\alpha_2} \in \langle w_1^{\beta_1} w_2^{\beta_2}, l_\tau^{\alpha_\tau} \rangle$ or $w_1^{\beta_1} w_2^{\beta_2} \in \langle u_1^{\alpha_1} u_2^{\alpha_2}, l_\tau^{\alpha_\tau} \rangle$ then $\text{reg}(C_\tau^\alpha(\Delta)) \leq M(\tau)$ is clear from Lemma 7.4. If $l_\tau^{\alpha_\tau} \in \langle u_1^{\alpha_1} u_2^{\alpha_2}, w_1^{\beta_1} w_2^{\beta_2} \rangle$ then $\alpha_\tau \geq \alpha_1 + \alpha_2, \alpha_\tau \geq \beta_1 + \beta_2$, and $\text{reg}(C_\tau^\alpha(\Delta)) \leq M(\tau)$ from Lemma 7.4. So we may assume $K(\tau)$ is minimally generated by the three given forms and $C_\tau^\alpha(\Delta) \cong \text{syz}(K(\tau))$. In this case $\text{reg}(C_\tau^\alpha(\Delta)) \leq \text{reg}(S/K(\tau)) + 2$ by two applications of Proposition 5.5 (equality holds but we will not need this). So it suffices to show that $\text{reg}(S/K(\tau)) \leq M(\tau) - 2$.

Four special cases are given by

- (1) $\alpha_1 = \beta_1 = 0 \implies K(\tau) = \langle l_\tau^{\alpha_\tau}, u_2^{\alpha_2}, w_2^{\beta_2} \rangle$
- (2) $\alpha_2 = \beta_2 = 0 \implies K(\tau) = \langle l_\tau^{\alpha_\tau}, u_1^{\alpha_1}, w_1^{\beta_1} \rangle$

$$(3) \quad \alpha_1 = \beta_2 = 0 \implies K(\tau) = \langle l_\tau^{\alpha_\tau}, u_2^{\alpha_2}, w_1^{\beta_1} \rangle$$

$$(4) \quad \alpha_2 = \beta_1 = 0 \implies K(\tau) = \langle l_\tau^{\alpha_\tau}, u_2^{\alpha_2}, w_1^{\beta_1} \rangle$$

Since $K(\tau)$ is minimally generated by the three given forms, [16, Theorem 2.7] applies in cases (1) and (2). For example, in case (1) we have

$$\begin{aligned} \text{reg}(S/K(\tau)) &= \left\lfloor \frac{\alpha_\tau + \alpha_2 + \beta_2 - 3}{2} \right\rfloor \\ &\leq \alpha_\tau + \alpha_2 + \beta_2 - 2 \\ &\leq M(\tau) - 2. \end{aligned}$$

A similar argument holds for case (2). In cases (3) and (4), $K(\tau)$ is a complete intersection of its generators and $\text{reg}(K(\tau)) \leq M(\tau) - 2$ follows from the Koszul resolution.

If at most one of $\alpha_1, \alpha_2, \beta_1, \beta_2$ vanishes we show $\text{reg}(S/K(\tau)) \leq M(\tau) - 2$ by fitting $S/K(\tau)$ into exact sequences and using Proposition 5.5. Let $Q = \langle l_\tau^{\alpha_\tau}, u_1^{\alpha_1} u_2^{\alpha_2} \rangle$. We have the short exact sequence

$$(4) \quad 0 \rightarrow \frac{S(-\beta_1 - \beta_2)}{Q : (w_1^{\beta_1} w_2^{\beta_2})} \xrightarrow{\cdot w_1^{\beta_1} w_2^{\beta_2}} \frac{S}{Q} \rightarrow \frac{S}{K(\tau)} \rightarrow 0$$

Q is a complete intersection with 2 generators in degrees α_τ and $\alpha_1 + \alpha_2$, so

$$\text{reg}(S/Q) = \alpha_\tau + \alpha_1 + \alpha_2 - 2$$

The ideal Q decomposes as $Q = \langle l_\tau^{\alpha_\tau}, u_1^{\alpha_1} \rangle \cap \langle l_\tau^{\alpha_\tau}, u_2^{\alpha_2} \rangle$. Then

$$Q : (w_1^{\beta_1} w_2^{\beta_2}) = I_1 \cap I_2,$$

where

$$\begin{aligned} I_1 &= \langle l_\tau^{\alpha_\tau}, u_1^{\alpha_1} \rangle : (w_1^{\beta_1} w_2^{\beta_2}) \\ I_2 &= \langle l_\tau^{\alpha_\tau}, u_2^{\alpha_2} \rangle : (w_1^{\beta_1} w_2^{\beta_2}) \end{aligned}$$

Since $(l_\tau^{\alpha_\tau}, u_1^{\beta_1})$ is (l_τ, u_1) -primary and $w_2 \notin (l_\tau, u_1)$,

$$I_1 = \langle l_\tau^{\alpha_\tau}, u_1^{\alpha_1} \rangle : (w_1^{\beta_1} w_2^{\beta_2}) = \langle l_\tau^{\alpha_\tau}, u_1^{\alpha_1} \rangle : w_1^{\beta_1}.$$

Similarly,

$$I_2 = \langle l_\tau^{\alpha_\tau}, u_2^{\alpha_2} \rangle : (w_1^{\beta_1} w_2^{\beta_2}) = \langle l_\tau^{\alpha_\tau}, u_2^{\alpha_2} \rangle : w_2^{\beta_2}.$$

From Proposition 7.5 below, if $I_1 \neq S$ and $I_2 \neq S$ then I_1, I_2 are complete intersections and

$$\begin{aligned} \text{reg}(S/I_1) &\leq \alpha_\tau + \alpha_1 - \beta_1 - 2 \\ \text{reg}(S/I_2) &\leq \alpha_\tau + \alpha_2 - \beta_2 - 2 \end{aligned}$$

We consider four final special cases before moving on to the general case.

$$\mathbf{A}: w_1^{\beta_1} \in \langle l_\tau^{\alpha_\tau}, u_1^{\alpha_1} \rangle \implies I_1 = S \implies Q : (w_1^{\beta_1} w_2^{\beta_2}) = I_2$$

$$\mathbf{B}: w_2^{\beta_2} \in \langle l_\tau^{\alpha_\tau}, u_2^{\alpha_2} \rangle \implies I_2 = S \implies Q : (w_1^{\beta_1} w_2^{\beta_2}) = I_1$$

Note that $\alpha_1 = 0$ falls under **A** and $\alpha_2 = 0$ falls under **B**. By the exact sequence (4) and Proposition 5.5 we have the corresponding bounds

$$\mathbf{A}: \text{reg}(S/K(\tau)) \leq \max\{\alpha_\tau + \alpha_2 + \beta_1 - 3, \alpha_\tau + \alpha_1 + \alpha_2 - 2\} \leq M(\tau) - 2$$

$$\mathbf{B}: \text{reg}(S/K(\tau)) \leq \max\{\alpha_\tau + \alpha_1 + \beta_2 - 3, \alpha_\tau + \alpha_1 + \alpha_2 - 2\} \leq M(\tau) - 2$$

If we use multiplication by $u_1^{\alpha_1} u_2^{\alpha_2}$ in the exact sequence (4) then we have the corresponding ideals $Q' = \langle l_\tau^{\alpha_\tau}, u_1^{\alpha_1} u_2^{\alpha_2} \rangle, I'_1 = \langle l_\tau^{\alpha_\tau}, w_1^{\beta_1} \rangle : u_1^{\alpha_1}$ and $I'_2 = \langle l_\tau^{\alpha_\tau}, w_2^{\beta_2} \rangle : u_2^{\alpha_2}$. We then have the analogous cases

$$\mathbf{C}: u_1^{\alpha_1} \in \langle l_\tau^{\alpha_\tau}, w_1^{\beta_1} \rangle \implies I'_1 = S \implies Q' : (u_1^{\alpha_1} u_2^{\alpha_2}) = I'_2$$

$$\mathbf{D}: u_2^{\alpha_2} \in \langle l_\tau^{\alpha_\tau}, w_2^{\beta_2} \rangle \implies I'_2 = S \implies Q' : (u_1^{\alpha_1} u_2^{\alpha_2}) = I'_1$$

Note that $\beta_1 = 0$ falls under \mathbf{C} and $\beta_2 = 0$ falls under \mathbf{D} . The corresponding bounds are

$$\mathbf{C}: \text{reg}(S/K(\tau)) \leq \max\{\alpha_\tau + \beta_2 + \alpha_1 - 3, \alpha_\tau + \beta_1 + \beta_2 - 2\} \leq M(\tau) - 2$$

$$\mathbf{D}: \text{reg}(S/K(\tau)) \leq \max\{\alpha_\tau + \beta_1 + \alpha_2 - 3, \alpha_\tau + \beta_1 + \beta_2 - 2\} \leq M(\tau) - 2.$$

We have reduced to the case where

- $w_i^{\beta_i} \notin \langle l_\tau^{\alpha_\tau}, u_i^{\alpha_i} \rangle$ (equivalently $\alpha_\tau + \alpha_i - \beta_i \geq 2$) for $i = 1, 2$
- $u_i^{\alpha_i} \notin \langle l_\tau^{\alpha_\tau}, w_i^{\beta_i} \rangle$ (equivalently $\alpha_\tau + \beta_i - \alpha_i \geq 2$) for $i = 1, 2$
- $\alpha_i \geq 1, \beta_i \geq 1$ for $i = 1, 2$ and $\alpha_\tau \geq 1$.

In particular, $u_1 \neq w_1$ implies that the vectors v_1, v_3, v_4 are linearly independent and $u_2 \neq w_2$ implies v_2, v_3, v_4 are linearly independent in Figure 4. It follows that we may make a change of coordinates so that v_1 points along the y -axis, v_2 points along the x -axis, and v_3 points along the z -axis. Applying appropriate scaling in the x , y , and positive z directions, we can assume that the vector defined by v_4 points in the direction of $\langle 1, 1, -1 \rangle$. Under this change of coordinates, $\text{st}(\tau)$ has four possible configurations, shown in Figure 5. The ideal $K(\tau)$ is the same for all of these. We have

$$\begin{aligned} l_\tau &= z \\ u_1 &= x \\ u_2 &= y \\ w_1 &= x + z \\ w_2 &= y + z \end{aligned}$$

and

$$\begin{aligned} I_1 &= \langle l_\tau^{\alpha_\tau}, u_1^{\alpha_1} \rangle : w_1^{\beta_1} = \langle x^{\alpha_1}, z^{\alpha_\tau} \rangle : (x + z)^{\beta_1} \\ I_2 &= \langle l_\tau^{\alpha_\tau}, u_2^{\alpha_2} \rangle : w_2^{\beta_2} = \langle y^{\alpha_2}, z^{\alpha_\tau} \rangle : (y + z)^{\beta_2} \end{aligned}$$

By Corollary 7.13 in the next section, $\text{reg}(S/Q) = \text{reg}(S/(I_1 \cap I_2)) \leq M(\tau) - \beta_1 - \beta_2 - 1$. By the exact sequence (4) and Lemma 5.5, the proof is complete. \square

7.1. Intersection of colon ideals. Let

$$\begin{aligned} I_1 &= \langle x^{\alpha_1}, z^{\alpha_\tau} \rangle : (x + z)^{\beta_1} \\ I_2 &= \langle y^{\alpha_2}, z^{\alpha_\tau} \rangle : (y + z)^{\beta_2} \end{aligned}$$

In [22], Tohaneanu and Minac compute the Hilbert function of the ideal (up to change of coordinates)

$$\langle x^{r+1}, (x + z)^{r+1} \rangle : z^{r+1} \cap \langle y^{r+1}, (y + z)^{r+1} \rangle : z^{r+1}.$$

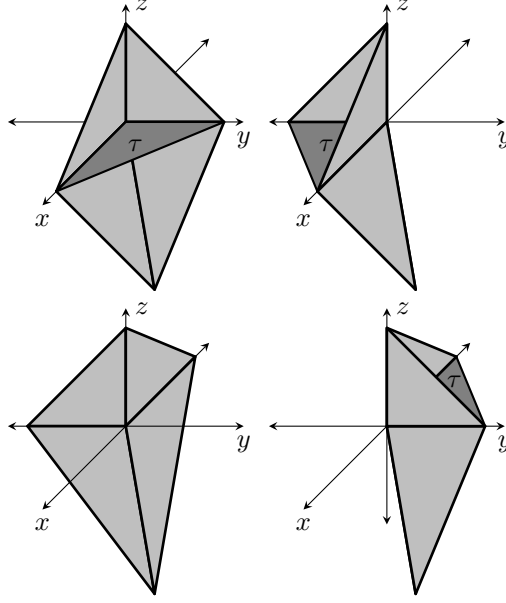
It is not so obvious how to apply their methods directly to the ideal $I_1 \cap I_2$. Building on their work, however, we show how to construct enough of the initial ideal (with respect to the lexicographic order) of $I_1 + I_2$ to give a fairly tight bound on the socle degree of $S/I_1 + I_2$. The methods synthesize descriptions of such ideals in terms of linear and commutative algebra.

We first compute the initial ideal of

$$I = I(p, q, r) = \langle s^p, t^q \rangle : (s + t)^r$$

in the ring $R = k[s, t]$ with standard lexicographic order. We assume $I \neq R$, so $(s + t)^r \notin \langle s^p, t^q \rangle$. This is equivalent to requiring $p + q - r \geq 2$.

Proposition 7.5. *Let $I = I(p, q, r) \subset R$ be as above, with $p + q - r \geq 2$. Then I is a complete intersection generated by two polynomials of*

FIGURE 5. Possible configurations for generic $\text{st}(\tau)$

- (1) degrees $a = \min\{p, q - r\}, b = \max\{p, q - r\}$ if $p + r - q \leq 1$.
- (2) degrees $a = \min\{q, p - r\}, b = \max\{q, p - r\}$ if $q + r - p \leq 1$.
- (3) degrees

$$a = \left\lfloor \frac{p + q - r}{2} \right\rfloor, b = \left\lceil \frac{p + q - r}{2} \right\rceil$$

if $p + r - q \geq 2$ and $q + r - p \geq 2$.

Proof. $p + r - q \leq 1$: In this case $t^q \in \langle s^p, (s + t)^r \rangle$. Let

$$t^q = f s^p + g(s + t)^r$$

for some polynomials $f, g \in R$, where g has no term divisible by s^p . It is immediate that

$$\langle s^p, t^q \rangle = \langle s^p, g(s + t)^r \rangle$$

and

$$I = \langle s^p, t^q \rangle : (s + t)^r = \langle s^p, g \rangle.$$

The polynomial g is not divisible by s since it has a term which is a constant multiple of t^{q-r} . It follows that s^p and g are relatively prime and I is a complete intersection. Since g has degree $q - r$, (1) is proved.

$q + r - p \leq 1$: The argument is identical to the previous case.

$p + r - q \geq 2$ and $q + r - p \geq 2$: Let

$$T = T(p, q, r) = \langle s^p, t^q, (s + t)^r \rangle.$$

Since we assume $p + q - r \geq 2$ as well, T is minimally generated by the three given generators. We describe I in terms of the minimal free resolution of the ideal T . Set

$a = \left\lfloor \frac{p + q - r}{2} \right\rfloor$ and $b = \left\lceil \frac{p + q - r}{2} \right\rceil$. The assumption $p + q - r \geq 2$ guarantees

that $a \geq 1$. T is a codimension two Cohen-Macaulay ideal with Hilbert-Burch resolution of the form below [16, Theorem 2.7]

$$0 \rightarrow R(-a-r) \oplus R(-b-r) \xrightarrow{\phi} R(-p) \oplus R(-q) \oplus R(-r) \rightarrow T$$

where

$$\phi = \begin{pmatrix} A & D \\ B & E \\ C & F \end{pmatrix}$$

is a matrix of forms satisfying $BF - EC = s^p$, $AF - DC = t^q$, $BF - EC = (s+t)^r$. It follows that the module of syzygies on T has two generators, corresponding to the relations

$$As^p + Bt^q + C(s+t)^r = 0$$

and

$$Ds^p + Et^q + F(s+t)^r = 0.$$

In terms of the entries of the matrix ϕ we may write

$$I = (C, F)$$

where $\deg(C) = a$, $\deg(F) = b$, and $a + b = p + q - r$. Since $AF - DC = t^q$ and $BF - EC = (s+t)^r$, any common factor of C and F would give a common factor of t and $(s+t)$, so C and F are relatively prime. So I is a complete intersection of the required degrees. \square

As an immediate corollary we have the following lemma.

Corollary 7.6. *With $I = I(p, q, r)$ as above, minimally generated by two forms of degree $a \leq b$, we have*

$$HF(I, d) = \binom{d+1-a}{1} + \binom{d+1-b}{1} - \binom{d+1-a-b}{1}.$$

Proof. From Proposition 7.5, I is a complete intersection of polynomials C, F with $\deg(C) = a$, $\deg(F) = b$. So I has minimal resolution of the form

$$0 \rightarrow R(-a-b) \rightarrow R(-a) \oplus R(-b) \rightarrow I \rightarrow 0.$$

The result follows from the additivity of Hilbert functions across exact sequences. \square

Given a Hilbert function $H(I, d)$, let L_d be the vector space spanned by the $H(I, d)$ greatest monomials of degree d with respect to lex order. Then the direct sum

$$L = \bigoplus_{d=0}^{\infty} L_d$$

is an ideal, known as the *lex-segment* ideal for the Hilbert function $H(d)$ [21, Proposition 2.21]. Since two *generic* forms in $\mathbb{R}[x, y]$ of degrees $a \leq b$ form a complete intersection, the ideal they generate has the same Hilbert function as I . Denote by $L(a, b)$ the corresponding lex-segment ideal. We will show that $L(a, b)$ is the initial ideal of $I(p, q, r)$.

To prove this we will use a matrix condition on the coefficients of a form f of degree d which distinguishes when $f \in I$. From Corollary 7.6, $I_d = R_d$ for

$d \geq a + b - 1$. Since $a + b = p + q - r$, this matrix condition we derive will be nontrivial for $1 \leq d < p + q - r - 1$. Suppose

$$f = \sum_{i+j=d} a_{i,j} s^i t^j$$

satisfies $f \in I_d$. Then by definition we have

$$f(s+t)^r \in (s^p, t^q)$$

Since the ideal on the right is a monomial ideal, $f \in I \iff$ every monomial of $f(s+t)^r$ is divisible by either s^p or t^q . Expanding $(s+t)^r$ and multiplying by f gives

$$f(s+t)^r = \sum_{i+j=d} \sum_{m+n=r} \binom{r}{m} a_{ij} s^{m+i} t^{n+j}$$

Setting $m+i=u$ and $n+j=v$ gives

$$\sum_{u+v=d+r} s^u t^v \left(\sum_{m+i=u} \binom{r}{m} a_{ij} \right).$$

$f \in I$ iff the only nonzero coefficients in this expression occur when $u \geq p$ or $v \geq q$. Since $v = d + r - u$, $v \geq q$ is equivalent to $u \leq d + r - q$. So $f \in I$ iff for $u = d + r - q + 1, \dots, p - 1$ we have the condition

$$\sum_{m+i=u} \binom{r}{m} a_{i,d-i} = 0.$$

Here we follow the convention that $\binom{A}{B} = 0$ when $B < 0$ or $B > A$. These fit together into the following matrix condition on the coefficients of f :

$$\begin{pmatrix} \binom{r}{d+r-q+1} & \binom{r}{d+r-q} & \binom{r}{d+r-q-1} & \cdots & \binom{r}{r-q+1} \\ \binom{r}{d+r-q+2} & \binom{r}{d+r-q+1} & \binom{r}{d+r-q} & \cdots & \binom{r}{r-q+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{r}{p} & \binom{r+1}{p-1} & \binom{r+1}{p-2} & \cdots & \binom{r}{p-d-2} \\ \binom{r}{p-1} & \binom{r}{p-2} & \binom{r}{p-3} & \cdots & \binom{r}{p-d-1} \end{pmatrix} \cdot \begin{pmatrix} a_{0,d} \\ a_{1,d-1} \\ \vdots \\ a_{d-1,1} \\ a_{d,0} \end{pmatrix} = 0.$$

Denote the $(p+q-r-d-1) \times (d+1)$ matrix on the left by $M(p, q, r, d)$. $M(p, q, r, d)$ has entries

$$M(p, q, r, d)_{i,j} = \binom{r}{d+r-q+1+i-j},$$

where $i = 0, \dots, \min\{p-1, p+q-r-d-2\}$ and $j = 0, \dots, d$. With this choice of indexing, column c_j of $M(p, q, r, d)$ corresponds to the coefficient $a_{j,d-j}$. The following lemma is fundamental for understanding $M(p, q, r, d)$.

Lemma 7.7. *Let $\mu = (\mu_0 \geq \dots \mu_k \geq 1)$ be a partition with $k+1$ parts so that $r \geq \mu_0$. Let $N(\mu)$ be the square matrix with entries*

$$N(\mu)_{ij} = \binom{r}{\mu_j + i - j}$$

for $i = 0, \dots, k$, $j = 0, \dots, k$. Then $N(\mu)$ has nonzero determinant.

Proof. This observation is made in [22, § 3.1], where it is noted that determinants of such matrices play a role in the representation theory of the special linear group $SL(V)$, where V is an r -dimensional vector space. In particular, if $\lambda = \mu'$, the conjugate partition to μ , $\det N(\mu)$ is the dimension of the Weyl module $S_\lambda V$, which is a nontrivial irreducible representation of $SL(V)$. More explicitly, $\det N(\mu) = s_\lambda(1, \dots, 1)$, where $s_\lambda(x_1, \dots, x_r)$ is the Schur polynomial in r variables of the partition $\lambda = \mu'$. In particular, $N(\mu)$ has nonzero determinant. See [15, § 6.1] and [15, Appendix A.1] for more details. \square

Corollary 7.8. *Let $M = M(p, q, r, d)$ be the $(p + q - r - d - 1) \times (d + 1)$ matrix defined as above, $M_{s,t}$ a nonzero entry of M , and k a nonnegative integer so that $s + k \leq p + q - r - d - 1$ and $t + k \leq d + 1$. Then*

- (1) *The $(k + 1) \times (k + 1)$ submatrix of M formed by the entries $\{M_{i,j} | s \leq i \leq s + k, t \leq j \leq t + k\}$ is invertible.*
- (2) *The rank of M is the minimum of the number of nonzero rows of M and the number of nonzero columns of M .*

Proof. (1) The submatrix of $M = M(p, q, r, d)$ above has entries

$$\binom{r}{d + r - q + 1 + s - t + i - j}$$

for $i = 0, \dots, k, j = 0, \dots, k$. Since we assume $M_{s,t} \neq 0$, $d + r - q + 1 + s - t \leq r$ and the first statement follows from Lemma 7.7 by taking $\mu_0 = \dots = \mu_k = d + r - q + 1 + s - t$.

(2) Observe that either $M_{0,0} \neq 0$ or, if $M_{0,0} = 0$, then the first entry $M_{j,0}$ ($j > 0$) which is nonzero is equal to 1. The last entry in the first column is $\binom{r}{p-1} \geq 1$ (we assumed $p \geq 1$), so there is at least one nonzero entry in the first column of M . If the row of M with index i is nonzero, every row with index $\geq i$ is also nonzero. If the column of M with index j is zero, every column with index $\geq j$ is also zero. Now the second statement follows by taking the largest square submatrix of M whose upper left corner is the first nonzero entry of the first column of M . This is a $k \times k$ submatrix of M where k is the minimum of the number of nonzero rows of M and the number of nonzero columns of M . By the first statement, this submatrix is invertible, and from the earlier observations k must be the rank of M . \square

Lemma 7.9. *The initial ideal of $I = I(p, q, r)$ is the lex-segment ideal $L(a, b)$, where $a \leq b$ are the degrees of the generators of I .*

Proof. For fixed degree d , let \mathfrak{r} be the rank of $M(p, q, r, d)$. By definition,

$$HF(I, d) = \dim \ker M(p, q, r, d),$$

hence $HF(I, d) = d + 1 - \mathfrak{r}$. From the submatrix constructed to prove part (2) of Corollary 7.8, the first \mathfrak{r} columns of M are linearly independent. It follows that for any column c_l of $M(p, q, r, d)$ with $\mathfrak{r} - 1 \leq l \leq d$, there is a unique (up to scaling) relation

$$\left(\sum_{i=0}^{\mathfrak{r}-1} a_{i,d-i} c_i \right) + a_{l,d-l} c_l = 0,$$

where $a_{l,d-l} \neq 0$. This gives rise to the polynomial $f = \sum_{i=0}^{p+q-r-d-2} a_{i,j} s^i t^{d-i} + a_{l,d-l} s^l t^{d-l} \in I$ with leading monomial $s^l t^{d-l}$. These monomials are the largest $d + 1 - \mathfrak{r}$ monomials of degree d with respect to lex ordering, so the result follows. \square

Corollary 7.10. *Let $I = (s^p, t^q) : (s + t)^r$, generated in degrees a and b , with $a \leq b$. The initial ideal of I with respect to the standard lexicographic order is*

$$L(a, b) = (s^a, s^{a-1}t^{b-a+1}, s^{a-2}t^{b-a+3}, \dots, s^{a-i}t^{b-a+2i-1}, \dots, t^{a+b-1}).$$

Proof. By Lemma 7.9 it suffices to show that the lex-segment ideal $L(a, b)$ has the form above. The Hilbert function of a complete intersection I generated in degrees a and b is

$$HF(I, d) = \binom{d+1-a}{1} + \binom{d+1-b}{1} - \binom{d+1-a-b}{1}.$$

More explicitly, we have

$$HF(I, d) = \begin{cases} 0 & \text{for } 0 \leq d < a \\ d+1-a & \text{for } a \leq d < b \\ 2d+2-(a+b) & \text{for } b \leq d \leq a+b-1 \\ d+1 & \text{for } d > a+b-1 \end{cases}$$

Recall $L(a, b)_d$ is the vector space spanned by the $HF(I, d)$ greatest monomials of degree d with respect to lex order. If $a \leq d < b$, then the $d+1-a$ greatest monomials are s^d, \dots, s^a . These are all divisible by s^a . If $b \leq d \leq a+b-1$, the $2d+2-(a+b)$ greatest monomials are $\{s^{d-i}t^i | i = 0, \dots, 2d-(a+b)+1\}$. If $i \leq d-a$, $s^{d-i}t^i$ is divisible by s^a . If $d-a \leq i \leq 2d+1-(a+b)$, then $s^{d-i}t^i = s^{a-j}t^{d-a+j} = s^{a-j}t^{b-a+(d-b+j)}$, where $j = 1, \dots, d-b+1$. This is divisible by $s^{a-j}t^{b-a+2j-1}$, which proves the corollary. \square

Remark 7.11. Conca and Valla [10] parametrize of all ideals in two variables with a given initial ideal. Using this, one can show that the lex-segment ideal $L(a, b)$ is the initial ideal of any ideal generated by two *generic* forms of degree a and b . Here *generic* means there are certain polynomials in the coefficients of the forms that must not vanish (the condition is not equivalent to the two forms being relatively prime). Lemma 7.9 can be viewed as a proof that the ideal I , which is generated by two forms, is generic in this sense.

Proposition 7.12. *Set $R = k[x, y]$, $S = k[x, y, z]$ both with standard lexicographic orders. For positive integers $a \leq b$, $c \leq d$, let $J_1, J_2 \subset R = k[s, t]$ be ideals satisfying $\text{in}(J_1) = L(a, b)$ and $\text{in}(J_2) = L(c, d)$, respectively. Let $S = k[x, y, z]$ and define ring maps $i_1, i_2 : R \rightarrow S$ by $i_1(s) = x, i_1(t) = z$ and $i_2(s) = y, i_2(t) = z$. Set $I_1 = i_1(J_1)S$, $I_2 = i_2(J_2)S$, and $N = \max\{a+d-1, b+c-1\}$. Then*

$$(I_1 + I_2)_N = S_N$$

Proof. It suffices to show that $(\text{in}(I_1) + \text{in}(I_2))_N = S_N$. We have

$$\begin{aligned} \text{in}(I_1) &= \langle x^a \rangle + \langle x^{a-i}z^{b-a+2i-1} | i = 1, \dots, a \rangle \\ \text{in}(I_2) &= \langle y^c \rangle + \langle y^{c-j}z^{d-c+2j-1} | j = 1, \dots, c \rangle \end{aligned}$$

Let $m = x^i y^j z^k$ be a monomial of S with degree N . We claim $m \in \text{in}(I_1) + \text{in}(I_2)$. If $i \geq a$ or $j \geq c$ then $x^a | m$ or $y^c | m$ and we are done. So set $i = a-s, j = c-t$, where $1 \leq s \leq a, 1 \leq t \leq c$. If $s \leq t$ then $a+c-s-t+(b-a+2s-1) = b+c-1+s-t \leq N$. So $k = N-(a+c-s-t) \geq b-a+2s-1$ and $x^i y^j z^k \in \text{in}(I_1)$. If $t \leq s$ then $a+c-s-t+(d-c+2t-1) = a+d-s+t-1 \leq N$. So $k = N-(a+c-s-t) \geq d-c+2t-1$ and $x^i y^j z^k \in \text{in}(I_2)$. \square

Corollary 7.13. *Let*

$$\begin{aligned} I_1 &= \langle x^{\alpha_1}, z^{\alpha_\tau} \rangle : (x+z)^{\beta_1} \\ I_2 &= \langle y^{\alpha_2}, z^{\alpha_\tau} \rangle : (y+z)^{\beta_2} \end{aligned}$$

where $\alpha_i + \alpha_\tau - \beta_i \geq 2$ and $\beta_i + \alpha_\tau - \alpha_i \geq 2$ for $i = 1, 2$. Also assume $\alpha_i \geq 1, \beta_i \geq 1$ for $i = 1, 2$ and $\alpha_\tau \geq 1$. Let

$$\begin{aligned} M(\tau) = & \alpha_\tau + \max\{\alpha_1 + \alpha_2, \alpha_1 + \beta_1, \alpha_1 + \beta_2, \alpha_2 + \beta_1, \\ & \alpha_2 + \beta_2, \beta_1 + \beta_2, \alpha_\tau + \alpha_1, \alpha_\tau + \alpha_2, \alpha_\tau + \beta_1, \alpha_\tau + \beta_2\} \end{aligned}$$

as in the statement of Theorem 7.1. Then

$$\operatorname{reg}\left(\frac{S}{I_1 \cap I_2}\right) \leq M(\tau) - \beta_1 - \beta_2 - 1$$

Proof. We use the short exact sequence

$$0 \rightarrow \frac{S}{I_1 \cap I_2} \rightarrow \frac{S}{I_1} \oplus \frac{S}{I_2} \rightarrow \frac{S}{I_1 + I_2} \rightarrow 0$$

and Proposition 5.5. From Proposition 7.5, $\operatorname{reg}(S/I_1) = \alpha_1 + \alpha_\tau - \beta_1 - 2 \leq M(\tau) - \beta_1 - \beta_2 - 1$ and $\operatorname{reg}(S/I_2) = \alpha_2 + \alpha_\tau - \beta_2 - 2 \leq M(\tau) - \beta_1 - \beta_2 - 1$. We show $\operatorname{reg}(S/(I_1 + I_2)) \leq M(\tau) - \beta_1 - \beta_2 - 2$; then we are done by Proposition 5.5. Equivalently, we show $(I_1 + I_2)_d = S_d$ for $d = M(\tau) - \beta_1 - \beta_2 - 1$. Let I_1, I_2 be generated in degrees $a \leq b, c \leq d$ respectively. By Lemma 7.9 and Proposition 7.12, $(I_1 + I_2)_d = S_d$ for $d \geq \max\{a + d - 1, b + c - 1\}$. So we need to show that $\max\{a + d, b + c\} \leq M(\tau) - \beta_1 - \beta_2$. We consider 4 cases.

- (1) $\beta_i + \alpha_i - \alpha_\tau \geq 2$ for $i = 1, 2$.
- (2) $\beta_1 + \alpha_1 - \alpha_\tau \leq 1$ and $\beta_2 + \alpha_2 - \alpha_\tau \geq 2$.
- (3) $\beta_2 + \alpha_2 - \alpha_\tau \geq 2$ and $\beta_1 + \alpha_1 - \alpha_\tau \leq 1$.
- (4) $\beta_i + \alpha_i - \alpha_\tau \leq 1$ for $i = 1, 2$.

Case 1: By Proposition 7.5, $b - a \leq 1$ and $d - c \leq 1$. Suppose $b < d$. Then $a \leq c$, so $b + c < d + c$ and $a + d \leq c + d$, where $c + d = \alpha_2 + \alpha_\tau - \beta_2 \leq M(\tau) - \beta_1 - \beta_2$. Similarly if $d < b$ then $b + c \leq b + a$ and $a + d < a + b$, where $a + b = \alpha_1 + \alpha_\tau - \beta_1 \leq M(\tau) - \beta_1 - \beta_2$. If $b = d$ then $a + d = a + b \leq M(\tau) - \beta_1 - \beta_2$ and $b + c = d + c \leq M(\tau) - \beta_1 - \beta_2$. Hence $\max\{a + d - 1, b + c - 1\} \leq M(\tau) - \beta_1 - \beta_2$.

Case 2: By Proposition 7.5, $a = \min\{\alpha_1, \alpha_\tau - \beta_1\}$ and $b = \max\{\alpha_1, \alpha_\tau - \beta_1\}$. Since $\alpha_1 \leq \alpha_\tau - \beta_1 + 1$ by assumption, $a \leq \alpha_\tau - \beta_1$ and $b \leq \alpha_\tau - \beta_1 + 1$. By Proposition 7.5,

$$\begin{aligned} c &= \left\lfloor \frac{\alpha_2 + \alpha_\tau - \beta_2}{2} \right\rfloor \\ d &= \left\lceil \frac{\alpha_2 + \alpha_\tau - \beta_2}{2} \right\rceil. \end{aligned}$$

Hence

$$\max\{a + d, b + c\} \leq \alpha_\tau - \beta_1 + 1 + \left\lfloor \frac{\alpha_2 + \alpha_\tau - \beta_2}{2} \right\rfloor.$$

$\alpha_2 + \alpha_\tau - \beta_2 \geq 2$, so

$$\left\lfloor \frac{\alpha_2 + \alpha_\tau - \beta_2}{2} \right\rfloor \leq \alpha_2 + \alpha_\tau - \beta_2 - 1.$$

It follows that

$$\begin{aligned} \max\{a + d, b + c\} &\leq \alpha_\tau + \alpha_2 + \alpha_\tau - \beta_1 - \beta_2 \\ &\leq M(\tau) - \beta_1 - \beta_2. \end{aligned}$$

Case 3: By arguing exactly as in Case 2 we obtain

$$\begin{aligned} \max\{a + d, b + c\} &\leq \alpha_\tau + \alpha_1 + \alpha_\tau - \beta_1 - \beta_2 \\ &\leq M(\tau) - \beta_1 - \beta_2. \end{aligned}$$

Case 4: By Proposition 7.5, I_1 is generated in degrees $a = \min\{\alpha_1, \alpha_\tau - \beta_1\}$, $b = \max\{\alpha_1, \alpha_\tau - \beta_1\}$ and I_2 is generated in degrees $c = \min\{\alpha_2, \alpha_\tau - \beta_2\}$, $d = \max\{\alpha_2, \alpha_\tau - \beta_2\}$. We have $\alpha_1 \leq \alpha_\tau - \beta_1 + 1$ and $\alpha_2 \leq \alpha_\tau - \beta_2 + 1$ by assumption. It follows that $a \leq \alpha_\tau - \beta_1$, $b \leq \alpha_\tau - \beta_1 + 1$ and $c \leq \alpha_\tau - \beta_2$, $d \leq \alpha_\tau - \beta_2 + 1$. So

$$\begin{aligned} \max\{a + d, b + c\} &\leq \alpha_\tau + \alpha_\tau + 1 - \beta_1 - \beta_2 \\ &\leq M(\tau) - \beta_1 - \beta_2. \end{aligned}$$

The final inequality follows since we assumed $\alpha_1, \alpha_2, \beta_1, \beta_2, \alpha_\tau$ are all at least 1. \square

8. EXAMPLES

We give several examples to illustrate both how the bounds in Theorems 6.7 and 7.2 may be used and how well they approximate the actual regularity of the spline algebra. These examples also elucidate a difference between *complete* central complexes \mathcal{P} (in which the intersection of all facets of \mathcal{P} is an *interior* face of \mathcal{P}) and central complexes which are not complete. This difference is key to Conjecture 9.1 in the following section.

Example 8.1. In this example we apply Theorem 6.7 to bound the regularity of $C^\alpha(\mathcal{P})$ where boundary vanishing is imposed. Consider the two dimensional polytopal complex \mathcal{Q} in Figure 6 with five faces, eight interior edges, and four interior vertices. Impose vanishing of order r along interior codimension one edges

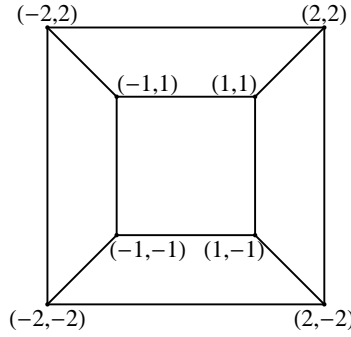


FIGURE 6. \mathcal{Q}

and vanishing of order s along boundary codimension one faces. The following Hilbert polynomials are computed in [12, Example 8.5]. If $s = -1$, then

$$\begin{aligned} HP(C^\alpha(\widehat{\mathcal{P}}), d) = & \frac{5}{2}d^2 + \left(-8r - \frac{1}{2}\right)d \\ & - 4 \left\lfloor \frac{3r}{2} \right\rfloor^2 + 12r \left\lfloor \frac{3r}{2} \right\rfloor - r^2 + 4r + 2 \end{aligned}$$

By Theorem 6.7, $\text{reg}C^r(\widehat{\mathcal{Q}}) \leq 6(r+1) - 1$ and $HP(C^r(\widehat{\mathcal{Q}}), d) = \dim C_d^r(\mathcal{Q})$ for $d \geq 6(r+1) - 2$. We compare the regularity bound $6(r+1) - 1$ with $\text{reg}C^r(\widehat{\mathcal{Q}})$ as computed in Macaulay2 in Table 2. $\text{reg}C^r(\widehat{\mathcal{Q}})$ appears to have alternating differences of 1 and 3 and grows roughly as $2(r+1) + 1$. In fact $\text{reg}C^r(\widehat{\mathcal{Q}})$ appears to agree with the regularity of r -splines on the complex from Example 1.1.

r	0	1	2	3	4	5	6	7	8	9
$6(r+1) - 1$	5	11	17	23	29	35	41	47	53	59
$\text{reg}(C^r(\widehat{\mathcal{Q}}))$	3	4	7	8	11	12	15	16	19	20

TABLE 2

Now suppose that vanishing of degree $s \geq 0$ is imposed along $\partial\mathcal{P}$. Then

$$\begin{aligned}
HP(C^\alpha(\widehat{\mathcal{P}}), d) = & \frac{5}{2}d^2 + (-8r - 4s - \frac{9}{2})d \\
& -3 \left\lfloor \frac{2(r+s)}{3} \right\rfloor^2 + 4r \left\lfloor \frac{2(r+s)}{3} \right\rfloor + 4s \left\lfloor \frac{2(r+s)}{3} \right\rfloor - \left\lfloor \frac{2(r+s)}{3} \right\rfloor \\
& -4 \left\lfloor \frac{r}{2} \right\rfloor^2 - 4 \left\lfloor \frac{3r}{2} \right\rfloor^2 + 4r \left\lfloor \frac{r}{2} \right\rfloor + 12r \left\lfloor \frac{3r}{2} \right\rfloor \\
& -5r^2 + 4rs + 8r + 4s + 4.
\end{aligned}$$

This formula is correct when r, s are not too small; for instance if $r = 3$ and $s = 0$, the above formula has constant term 81 while the actual constant, according to Macaulay2, is 87. By Theorem 6.7,

$$\text{reg}(C^\alpha(\widehat{\mathcal{Q}})) \leq \max\{6(r+1) + (s+1), 5(r+1) + 2(s+1)\} - 1$$

and $HP(C^\alpha(\widehat{\mathcal{P}}), d) = \dim C_d^\alpha(\mathcal{P})$ for

$$d \geq \max\{6(r+1) + (s+1), 5(r+1) + 2(s+1)\} - 2.$$

A comparison of the bound on $\text{reg}(C^\alpha(\widehat{\mathcal{Q}}))$ and its actual value computed in Macaulay2 appears in Table 3 for $r, s \leq 5$.

	$\max\{6(r+1) + (s+1), 5(r+1) + 2(s+1)\} - 1$				
	$s = 0$	$s = 1$	$s = 2$	$s = 3$	$s = 4$
$r = 0$	7	8	10	12	14
$r = 1$	13	14	15	17	19
$r = 2$	19	20	21	22	24
$r = 3$	25	26	27	28	29
$r = 4$	31	32	33	34	35

	$\text{reg}(C^\alpha(\widehat{\mathcal{Q}}))$				
	$s = 0$	$s = 1$	$s = 2$	$s = 3$	$s = 4$
$r = 0$	4	4	5	6	7
$r = 1$	4	5	6	8	9
$r = 2$	7	8	8	9	10
$r = 3$	8	8	9	10	12
$r = 4$	11	11	12	12	13

TABLE 3

Example 8.2. We now give an example which has very different behavior from Example 5.9. Consider the two-dimensional polytopal complex \mathcal{Q} formed by placing a regular (or almost regular) n -gon inside of a scaled copy of itself and connecting corresponding vertices by edges. \mathcal{Q} has one facet with n edges and n quadrilateral facets. An example for $n = 10$ is shown in Figure 7. We may or may not perturb

the vertices so that the affine spans of the edges between the inner and outer n -gons do not all meet at the origin. This does not appear to have much effect on regularity, although it does change the constant term of $HP(C^r(\hat{\mathcal{Q}}), d)$.

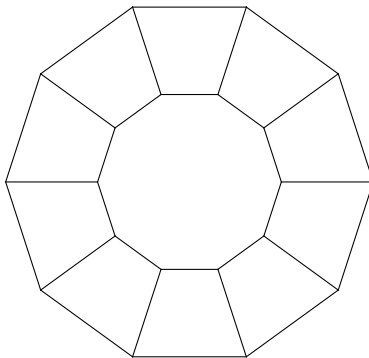


FIGURE 7

According to Theorem 6.7, $\text{reg}(C^r(\hat{\mathcal{Q}})) \leq \max\{(r+1)(n+2), 5(r+1)\} \leq (r+1)(n+2)$ as long as $n \geq 3$. However, according to computations for $r \leq 3$ and $n \leq 10$ in Macaulay2, $\text{reg}(\mathcal{Q}) \leq 3(r+1)$ regardless of what value n takes. It appears that having a facet σ with many codimension one facets may only significantly effect the regularity of $C^\alpha(\mathcal{P})$ if $\sigma \cap \partial\mathcal{P} \neq \emptyset$, as in Example 5.9.

Example 8.3. Consider a regular octahedron $\Delta \subset \mathbb{R}^3$ triangulated by placing a centrally symmetric vertex, shown in Figure 8. In [25, Example 5.2], Schenck shows that $C^r(\Delta)$ is free, generated in degrees $r+1$, $2(r+1)$, and $3(r+1)$. Thus the regularity bound for $C^r(\Delta)$ given by Corollary 6.3 is tight. Computations

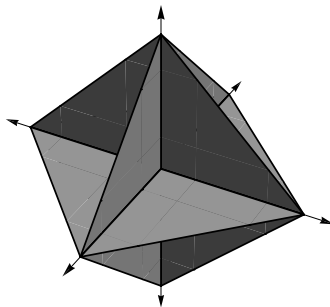


FIGURE 8. Centrally Triangulated Octahedron

in Macaulay2 suggest that the regularity of $C^r(\Delta)$ stays at $3(r+1)$ for generic perturbations of the noncentral vertices.

9. REGULARITY CONJECTURE

Let $\mathcal{P} \subset \mathbb{R}^n$ be a pure hereditary n -dimensional polytopal complex. We end with a conjecture in the case of uniform smoothness, where $\alpha(\tau) = r$ for all $\tau \in \mathcal{P}_{n-1}^0$

and $\alpha(\tau) = -1$ for $\tau \in (\partial\mathcal{P})_{n-1}$. This conjecture is a slight refinement of [11, Conjecture 5.6]. For $\sigma \in \mathcal{P}_n$, recall $|\partial^0(\sigma)|$ is the number of codimension one faces of σ which are interior to the complex. We will call a central complex \mathcal{P} *complete* if the intersection of all facets of \mathcal{P} is an *interior* face of \mathcal{P} .

Conjecture 9.1. *Let $\mathcal{P} \subset \mathbb{R}^{n+1}$ be a pure, central, hereditary n -dimensional polytopal complex. Call a facet $\sigma \in \mathcal{P}_{n+1}$ a boundary facet if it has a codimension one face $\tau \in \partial\mathcal{P}$ so that τ contains the cone vertex. Set*

$$\begin{aligned} F &= \max\{|\partial^0(\sigma)| : \sigma \in \mathcal{P}_{n+1}\} \\ F_{\partial} &= \max\{|\partial^0(\sigma)| : \sigma \in \mathcal{P}_{n+1} \text{ a boundary facet}\} \end{aligned}$$

Then,

- (1) *If \mathcal{P} is central and complete, then $\text{reg}(C^r(\mathcal{P})) \leq \text{reg}(LS^{r,n-1}(\mathcal{P})) \leq F(r+1)$ and this bound is tight.*
- (2) *If \mathcal{P} is central but not complete, then $\text{reg}(C^r(\mathcal{P})) \leq F_{\partial}(r+1)$.*

Furthermore, the bound is attained by free modules $C^r(\mathcal{P})$ in both cases.

Remark 9.2. Example 5.9 shows that generators can be obtained in degree $F(r+1)$ for the complete central case and degree $F_{\partial}(r+1)$ in the non-complete central case, so these are the lowest possible regularity bounds that we can conjecture.

Remark 9.3. If $C^r(\mathcal{P})$ is free, then $\text{reg}(C^r(\mathcal{P})) \leq F(r+1)$ by Corollary 6.3. Example 8.3, coupled with Theorem 7.2, shows that Conjecture 9.1 is true in the complete, central, three dimensional, simplicial case. If $\mathcal{P} \subset \mathbb{R}^3$ is complete, central, and non-simplicial, then Conjecture 9.1 should be provable using the methods of § 7. The difficulty is in analyzing the ideal $K(\tau)$ from Lemma 7.4.

Remark 9.4. Conjecture 9.1 part (2) is a natural generalization of a conjecture of Schenck [26], that $\text{reg}(C^r(\hat{\Delta})) \leq 2(r+1)$ for $\Delta \subset \mathbb{R}^2$. This is a highly nontrivial conjecture in the simplicial case; it implies, for instance, that $\varphi(C^1(\hat{\Delta})) \leq 2$. To date, it is unknown whether $HP(C^1(\hat{\Delta}), 3) = \dim C_3^1(\hat{\Delta})$. The difficulty of this problem is in large part due to the fact that non-local geometry plays an increasingly important role in low degree [1, 5]. Since our methods hinge on using the algebras $LS^{\alpha,k}(\mathcal{P})$, which are locally supported approximations to $C^{\alpha}(\mathcal{P})$, our approach will not be effective in proving Conjecture 9.1 part (2).

Remark 9.5. In the non-simplicial case, Conjecture 9.1 part (2) appears to run contrary to the spirit of the regularity bounds we have proved in this paper, since no account is taken of interior facets of \mathcal{P} , which may have many codimension one faces. It is nevertheless consistent with Example 5.9, where the minimal generator of high degree is supported on a boundary facet, and Example 8.2, where an interior facet with many codimension one faces appears to have no contribution to $\text{reg}(C^r(\hat{\mathcal{P}}))$. An example of a polytopal complex \mathcal{P} with a minimal generator of high degree (relative to the number of codimension one faces of boundary facets), supported on interior facets, would be very interesting.

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